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综合论文训练（领军）

Langlands duality of the affine Hecke category

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摘　要

设 k 为特征 $p>0$ 的有限域的代数闭包，G 为 k 上的分裂约化群。受仿射 Hecke 代数的几何实现及其在具有幺幂单值的局部 Langlands 对应中的作用启发，本文研究反球面模 (anti-spherical module) 与对偶 Springer 消解的等变 K-理论之间等式的范畴化。沿用 Arkhipov–Bezrukavnikov 的框架，并结合 Achar–Riche 的相关发展，本文回顾并使用 Gaitsgory 的中心函子、Wakimoto 层及其滤过，以及仿射旗簇上的 Iwahori–Whittaker 层范畴。借助这些结构、Tannakian 形式化以及去等变化 (de-equivariantization) 方法，本文构造从 $\widetilde{\mathcal{N}}^{\vee}$ 上的凝聚层范畴到仿射旗簇上层范畴的相关函子，并通过 Iwahori–Whittaker 范畴重新得到 Arkhipov–Bezrukavnikov 等价。本文的主要结果是该等价的分次版本：利用分次层理论，构造并证明如下张量范畴等价

$$\Phi_{\mathrm{gr},\mathcal{IW}} : D\mathrm{Coh}^{G^{\vee}\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^{\vee}\right) \simeq D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

由此，本文给出了

$$\mathcal{M}^{\mathrm{asph}} = K^{G^{\vee}\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^{\vee}\right)$$

的完整范畴化，并保留了 $\mathbb{G}_m$-等变结构所对应的分次信息。

关键词：范畴化；Arkhipov–Bezrukavnikov 等价；去等变化

Abstract

Let k be an algebraic closure of a finite field of characteristic $p > 0$, and let G be a split reductive group over k. Motivated by the geometric realization of affine Hecke algebras and their role in the local Langlands correspondence with unipotent monodromy, this paper studies a categorical refinement of the equivalence between the antispherical module and the equivariant K-theory of the Springer resolution. Following the framework of Arkhipov–Bezrukavnikov and related developments of Achar–Riche, we review Gaitsgory's central functor, Wakimoto sheaves and their filtrations, and the category of Iwahori–Whittaker sheaves on the affine flag variety. Using these structures, together with Tannakian formalism and de-equivariantization arguments, we construct the relevant functors from coherent sheaves on $\widetilde{\mathcal{N}}^{\vee}$ to sheaf categories on the affine flag variety and recover the Arkhipov–Bezrukavnikov equivalence through the Iwahori–Whittaker category. The main result is a graded refinement: using the theory of graded sheaves, we construct and prove the monoidal equivalence

$$\Phi_{\mathrm{gr},\mathcal{IW}} : D\mathrm{Coh}^{G^{\vee}\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^{\vee}\right) \simeq D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

This gives a complete categorification of the equivalence

$$\mathcal{M}^{\mathrm{asph}} = K^{G^{\vee}\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^{\vee}\right)$$

retaining the $\mathbb{G}_m$-equivariant grading.

Keywords: Categorification; Arkhipov–Bezrukavnikov equivalence; de-equivariantization

Contents

Chapter 1 Introduction

1.1. Historical remarks

The whole story begins with the local Langlands correspondence.

Conjecture 1.1.1 (Local Langlands correspondence). We fix a local field K and a split reductive group G over K. Then there exists a canonical finite-to-one map

$$\left\{\begin{matrix}\text{Irreducible smooth admissible}\\ \text{representations of } G(K) \text{ over } \mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Frobenius semisimple Weil–}\\ \text{Deligne representations } W_K \to G^\vee(\mathbb{C})\end{matrix}\right\}$$

This correspondence is very difficult, and is fully established only in the case of GL_n. However, one may restrict to unipotent monodromy, and the correspondence takes the following form

Conjecture 1.1.2 (Local Langlands correspondence with unipotent monodromy). There exists a canonical finite-to-one map

$$\left\{\begin{matrix}\text{Representations of } G_k\\ \text{with a non-zero } I\text{-fixed vector}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Representations of } W_K\\ \text{factoring through } W_K \twoheadrightarrow \mathbb{Z} \to G^\vee(\mathbb{C})\end{matrix}\right\}$$

Then note that the tamely ramified representations with unipotent monodromy coincide with modules over the Hecke algebra $\mathcal{H}(G(K), I)$. This is equivalently our affine Hecke algebra $\mathcal{H}$ base changed to $\mathbb{C}$, $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$. Deligne–Langlands conjectured the following.

Conjecture 1.1.3 (Deligne–Langlands). Irreducible $\mathcal{H}(G(K), I)$-representations are parametrized by $G^\vee$-conjugacy classes of triples (s, N, ρ), where $s \in G^\vee$ is a semisimple element, $N \in \mathcal{N}^\vee$ is a nilpotent element in the dual Lie algebra such that $\mathrm{Ad}(s)N = qN$, and ρ is a representation of $\pi_0(Z_{G^\vee}(s, N))$.

This is basically a translation of the local Langlands correspondence with unipotent monodromy.

Then, Kazhdan–Lusztig proved the following geometric interpretation of the affine Hecke algebra.

Theorem 1.1.4 (Kazhdan–Lusztig[1]). *We have the following equivalence of algebras*

$$\mathcal{H} = K^{G^\vee \times \mathbb{G}_m}(\mathrm{St}^\vee)$$

where $\mathrm{St}^\vee$ *is the dual Steinberg variety* $\mathrm{St}^\vee = \widetilde{\mathcal{N}}^\vee \times_{\mathcal{N}^\vee} \widetilde{\mathcal{N}}^\vee$ *and* $\mathcal{H}$ *is the affine Hecke algebra.*

Using this interpretation, they were able to establish the Deligne–Langlands conjecture. The construction is as follows: let (s, N, ρ) be a triple, and consider $K(\mathcal{B}_N^s)$, where $\mathcal{B}_N^s$ denotes the s-fixed locus in the Springer fiber over N. This is a module over $K^{G^\vee \times \mathbb{G}_m}(\mathrm{St}^\vee)$. Then $\pi_0(Z_{G^\vee}(s, N))$ acts on this module, yielding a decomposition into isotypic components. Taking an irreducible quotient yields the desired $\mathcal{H}$-module. This is how the tamely ramified, unipotent-monodromy case of the local Langlands correspondence for split reductive groups is solved.

Then, Chriss–Ginzburg[2] refined Kazhdan–Lusztig's proof. A central feature of the refinement is that they used the following equivalence

$$\mathcal{M}^{\mathrm{asph}} = K^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right)$$

and the fact that these are faithful modules over the corresponding algebras to establish the equivalence of algebras.

At almost the same time, people discovered the K-theoretic interpretation of the Hecke algebra; see Proposition 2.5.6, essentially first established by Kazhdan–Lusztig[3]. Thus, the above equivalence of algebras becomes an equivalence of K-groups, which suggests a categorification. This was done by Bezrukavnikov[4], although without the $\mathbb{G}_m$ -action, in the following form.

Theorem 1.1.5 (Bezrukavnikov). *There is a monoidal equivalence of categories*

$$\Phi_{I,I} : D(I\backslash LG/I) \to D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee \times^L_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee\right)$$

where $\widetilde{\mathcal{N}}^\vee \times^L_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee$ *is a derived fiber product, and this* $D\mathrm{Coh}^{G^\vee}$ *means its category of coherent sheaves (naturally in the derived sense).*

What's more, he proved the monodromic version of the above equivalence, namely

Theorem 1.1.6 (Bezrukavnikov). *There is a monoidal equivalence of categories*

$$\Phi_{I_0,I_0} : D_{\mathrm{mon}}(I_0\backslash LG/I_0) \to D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee \times^L_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee\right)$$

where D^{mon} denotes the category of unipotently monodromic sheaves.

This can be regarded as the unipotently ramified version of the geometric Langlands program. Moreover, this result inspired Dhillon–Taylor for their work on the Betti setting.

Theorem 1.1.7 (Dhillon–Taylor[5]). *We have an equivalence of monoidal ∞-categories*

$$\mathrm{IndCoh}_{G^\vee}\left(\tilde{G}^\vee \times_{G^\vee} \tilde{G}^\vee\right) \simeq \mathrm{Shv}_{\mathrm{nilp}}(I_0\backslash LG/I_0)$$

1.2. Main results

We let k be the algebraic closure of a finite field of characteristic $p > 0$, and assume that G is a split reductive group over k.

We aim to categorify the central step in proving the above conjecture, namely,

$$\mathcal{M}^{\mathrm{asph}} = K^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right)$$

so we follow Arkhipov–Bezrukavnikov[6] and Achar–Riche[7], and recall the proof of the following equivalence:

Theorem 1.2.1 (Main theorem[6]). *There is a monoidal equivalence of categories*

$$\Phi : D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \simeq D\left({}^f\mathcal{P}_I\right)$$

although the version Theorem 2.4.1 is considered the more fundamental one.

The main technical difficulties of the paper[6] are the construction of certain sheaves that serve as generators for the right-hand-side category and a version of Tannakian formalism used to construct the functor and prove the equivalence.

In the second part, we repeat the main result in the graded setting. Namely, we prove the following conjectural statement.

Conjecture 1.2.2. There is a monoidal equivalence of categories

$$\Phi_{\mathcal{IW},\mathrm{gr}} : D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \simeq D_{\mathcal{IW},\mathrm{gr}}(\mathrm{Fl})$$

where $D_{\mathrm{gr}}(\mathrm{Fl})$ is the category of graded sheaves introduced by Ho–Li[8].

In this paper, we will prove this conjecture, hence give a complete categorification of the equivalence $\mathcal{M}^{\mathrm{asph}} = K^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right)$.

1.3. Conventions

Involving derived categories:

(1) All derived categories (or homotopy categories) will mean bounded derived (homotopy) categories.

(2) We will not consider the stable ∞-structures of most derived categories, with the exception that when we will define the category of graded sheaves. Instead, we will work with the underlying triangulated categories.

Involving ℓ-adic sheaves:

(1) For X a stack of finite type over k or $\mathbb{F}_q$, We denote $D(X)$ by its bounded derived category of constructible ℓ-adic sheaves.

(2) As usual, we follow Zhu[9] for the following convention. For ind-schemes such as the affine flag variety, we define its derived category of constructible ℓ-adic sheaves via taking the colimit, namely, for $X = \operatorname{colim} X_i$ a presentation of an ind-scheme by its closed subschemes, we define

$$D(X) = \operatorname{colim} D(X_i)$$

(3) If moreover in the previous setting, there is a pro-algebraic group K acting on the ind-scheme X with nice conditions, we can fix a presentation of $X = \operatorname{colim} X_i$ such that X_i is K-stable, and there is a finite type quotient $K \twoheadrightarrow K_i$ such that K acts on X_i through K_i. Then, we may define

$$D_K(X) = \operatorname{colim} D_{K_i}(X_i)$$

(4) For any stack X over k with a model X_0 over $\mathbb{F}_q$, we abuse notation and write

$$D_{\text{mixed}}(X) \subset D(X_0)$$

to be the category of sheaves of integral Frobenius weights. This category is equipped with a Frobenius weight structure, given by the natural Weil structure on complexes of sheaves over X_0. Likewise, for a morphism $f : X \to Y$ coming from $f_0 : X_0 \to Y_0$, we define $f^* \dashv f_*$ and $f_! \dashv f^!$ using the corresponding operations on f_0. This causes a slight ambiguity, but it will not affect the final result, since all constructions involving the mixed category are independent of the chosen model, in particular the definition of graded sheaves. In particular, we assume our reductive group G has a split model over $\mathbb{F}_q$ for some q.

Miscellaneous:

(1) $\mathrm{Sh}(X)$ will denote any sheaf theory equipped with a nice enough six functor formalism, in particular in this paper, it will only mean the usual derived category of ℓ-adic sheaves, the graded ℓ-adic sheaves, or the mixed ℓ-adic sheaves.
(2) We will use cohomological t-indexing and homological weight indexing.
(3) Let A be an H-equivariant ring. We define $\mathbf{Mod}^H_{A,\mathrm{fr}}$ to be the modules of the form $V \otimes A^n$, where $V \in \mathbf{Rep}\, H$. Globally, let X be an H-equivariant scheme. We define $\mathrm{Coh}^H_{\mathrm{fr}}(X)$ to be the coherent sheaves of the form $V \otimes \mathcal{O}_X$.

Chapter 2 Sheaf theory

2.1. The affine flag variety and I-equivariant sheaves

Definition 2.1.1. Let $I \subset L^+G$ denote the Iwahori subgroup of LG. We define the affine flag variety to be $\mathrm{Fl} = LG/I$. Obviously, there is a projection map $p : LG \to LG/I$.

We define $D_I(\mathrm{Fl})$ to be the derived category of I-equivariant constructible ℓ-adic sheaves.

Construction 2.1.2. For a group G acting on schemes $G \circlearrowright Y$ and $X \circlearrowleft G$, and sheaves $\mathcal{F} \in \mathrm{Sh}(X/G)$, $\mathcal{G} \in \mathrm{Sh}(G\backslash Y)$, we may define the $\mathcal{F} \boxtimes_G \mathcal{G} \in \mathrm{Sh}\big(X \times^G Y\big)$ via the following process: consider the morphism

$$i : (X/G) \times (G\backslash Y) \to (X/G) \times_{\mathbb{B}G} (G\backslash Y) = X \times^G Y$$

and we define $\mathcal{F} \boxtimes_G \mathcal{G} = i^!(\mathcal{F} \boxtimes \mathcal{G})$.

Then, if we have some G-equivariant multiplication map $m : X \times Y \to Z$, we define the convolution product

$$\mathcal{F} *_G \mathcal{G} = m_*(\mathcal{F} \boxtimes_G \mathcal{G})$$

Such convolution product is naturally associative for three spaces able to do convolution.

In particular, such formalism will be compatible with the pro-group acting on ind-scheme formalism, since we may do the convolution for finite type groups on finite type schemes by passing to the presentation.

In our case, X, Y, Z will be sets of the form IwI or IwI/I, and the multiplication maps are given obviously. Moreover, m will be ind-proper, so we may define the convolution via $m_!$ rather than m_*, which we will use later on.

Construction 2.1.3. Recall we have the extended Weyl group $W^{\mathrm{ext}} = X_*(T) \rtimes W_f$, with the action of W_f on $X_*(T)$ is by the natural action of the Weyl group on the coweights. We have a length function making W^{ext} a quasi-coxeter group. Namely, we have it defined by $\ell : W^{\mathrm{ext}} \to \mathbb{N}$, given by for $w = xt_\lambda$ where $x \in W_f$ and $\lambda \in X_*(T)$

$$\ell(w) = \sum_{\substack{\alpha\in R^+ \\ x(\alpha)\in R^+}} |\langle\lambda,\alpha\rangle| + \sum_{\substack{\alpha\in R^+ \\ x(\alpha)\in R^-}} |\langle\lambda,\alpha\rangle + 1|$$

This gives W^{ext} the structure of a quasi-coxeter group. In particular, W^{ext} is generated by elements of length 1 and 0.

We let ${}^fW$ denote the set of minimal length representatives of the cosets W/W_f.

Then, we have the decomposition

$$LG = \coprod_{w\in W^{\text{ext}}} IwI = \coprod_{w\in W^{\text{ext}}} \mathrm{Fl}_w$$

and hence the I-orbits of Fl are indexed by W^{ext}. In particular, we have $\mathrm{Fl}_w = \mathbb{A}^{\ell(w)}$ (this is because the finite parts of I are affine). Moreover, we define $LG_w = p^{-1}(\mathrm{Fl}_w) = IwI$ where $p : LG \to \mathrm{Fl}$ denotes the projection.

Construction 2.1.4. We let $j_w : \mathrm{Fl}_w \to \mathrm{Fl}$ denote the inclusion of strata. We consider the following sheaves in $\mathbf{Perv}_I(\mathrm{Fl})$ (which we will denote by $\mathcal{P}_I$ for short)

$$j_{w!} = j_{w!}\big(\overline{\mathbb{Q}}_\ell[\ell(w)]\big), j_{w*} = j_{w*}\big(\overline{\mathbb{Q}}_\ell[\ell(w)]\big), L_w = j_{w!*}\big(\overline{\mathbb{Q}}_\ell[\ell(w)]\big)$$

where $\overline{\mathbb{Q}}_\ell[\ell(w)]$ is the shifted constant sheaf as a perverse sheaf on Fl_w. We define

$${}^f\mathcal{P}_I = \mathcal{P}_I/\operatorname{Span}\{L_w | w \notin {}^fW\}$$

We denote the projection $\mathcal{P}_I \to {}^f\mathcal{P}_I$ by $\mathcal{F} \mapsto {}^f\mathcal{F}$

Likewise, in the mixed setting, we have the sheaves in $\mathbf{Perv}_{I,\text{mixed}}(\mathrm{Fl})$

$$j^m_{w!} = j_{w!}\left(\overline{\mathbb{Q}}_\ell[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right), j^m_{w*} = j_{w*}\left(\overline{\mathbb{Q}}_\ell[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right)$$

We have the simple objects (as below) pure in this setting, because it is an intermediate extension of a pure sheaf.

$$L^m_w = j_{w!*}\left(\overline{\mathbb{Q}}_\ell[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right)$$

Note that in particular we have $j_{e!} = j_{e*} = \delta_e$, the skyscraper at the unit coset for $e \in W$ is the unit element.

We begin with some basic properties of these sheaves.

Lemma 2.1.5.

(1) For $w_1, w_2 \in W^{\mathrm{ext}}$ with $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, we have

$$j_{w_1!} *_I j_{w_2!} \simeq j_{w_1 w_2!}$$

and for $w_1, w_2, w_3 \in W^{\mathrm{ext}}$ such that $\ell(w_1 w_2 w_3) = \ell(w_1) + \ell(w_2) + \ell(w_3)$, the above isomorphism is compatible with the associativity of the convolution.

(2) The likewise result holds for j_{w*}.

(3) For any $w \in W^{\mathrm{ext}}$, there are canonical isomorphisms

$$j_{w!} *_I j_{w^{-1}*} \simeq \delta_e \simeq j_{w^{-1}*} *_I j_{w!}$$

Proof.

(1) By the formalism of Tits systems, for $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, we have the equation of subsets $Iw_1 I \cdot Iw_2 I = Iw_1 w_2 I$, and in particular, we have

$$LG_{w_1} \times^I \mathrm{Fl}_{w_2} = \mathrm{Fl}_{w_1 w_2}$$

then we have

$$j_{w_1 w_2!} = j_{w_1 w_2!}\left(\overline{\mathbb{Q}}_\ell\right)[\ell(w_1 w_2)] = j_{w_1 w_2!} m_!\left(\overline{\mathbb{Q}}_\ell[\ell(w_1)] \boxtimes_I \overline{\mathbb{Q}}_\ell[\ell(w_2)]\right) = j_{w_1!} *_I j_{w_2!}$$

(2) Similar to the previous proof.

(3) It suffices to prove the assertion for w a simple reflection and for w of length 0, by the associativity of the convolution product and the previous statements. To do this, we verify the stalks of the sheaf $j_{w!} *_I j_{w^{-1}*}$ vanishes outside I.

We compute the stalks of $j_{x!} *_I j_{y*}$. Since it is by definition I-equivariant, it suffices to compute its stalks at zI where $z \in W^{\mathrm{ext}}$. We temporarily forget the equivariance, and regard the convolution as a sheaf on Fl (rather than $I \backslash \mathrm{Fl}$). By base change theorem, we have

$$\left(j_{x!} *_I j_{y*}\right)_{zI} = m_!\left(j_{x!} \boxtimes_I j_{y*}\right)_{zI} = R\Gamma_c\left(m^{-1}(zI), \left(j_{x!} \boxtimes_I j_{y*}\right)|_{m^{-1}(zI)}\right)$$

We have an isomorphism $\mathrm{Fl} \to m^{-1}(zI)$ given by $gI \mapsto (g, g^{-1}zI)$. We consider the following diagram

$$
\begin{tikzcd}
& LG \times IyI \arrow[rr] \arrow[dd] & & LG \times \mathrm{Fl} \arrow[dd] \\
zIy^{-1}I \arrow[ur] \arrow[rr] \arrow[dd] & & LG \arrow[ur, "\tilde{f}"] \arrow[dd] & \\
& LG \times^I IyI \arrow[rr] & & LG \times^I \mathrm{Fl} \\
X \arrow[ur] \arrow[rr, "j_{z,y}"'] & & \mathrm{Fl} \arrow[ur, "f"'] &
\end{tikzcd}
$$

where the map $\tilde{f}$ is given by $g \mapsto (g, g^{-1}zI)$. Thus we may compute $\big(j_{x!} \boxtimes_I j_{y*}\big)|_{m^{-1}(zI)} = f^*\big(j_{x!} \boxtimes_I j_{y*}\big)$ as follows. We first forget the I-equivariant structure on all sheaves, which allows us to pass to the upper level of the above diagram. Then, we have

$$\tilde{f}^*\big(j_{x!} \boxtimes j_{y*}\big) = j_{x!} \otimes f_2^* j_{y*} = j_{x!} \otimes \tilde{j}_{z,y*}\overline{\mathbb{Q}}_\ell[\ell(y)]$$

which is because $\tilde{j}_{z,y} : zIy^{-1}I \hookrightarrow LG$ is precisely the pullback of $\tilde{j}_y : IyI \hookrightarrow LG$ along the lift of $\tilde{f}_2$, namely $g \mapsto g^{-1}z$. Note that this is an isomorphism, so $*$-pushforward commutes with the corresponding $*$-pullbacks. Thus, we have the formula

$$f^*\big(j_{x!} \boxtimes_I j_{y*}\big) = j_{x!} \otimes j_{z,y*}\overline{\mathbb{Q}}_\ell[\ell(y)]$$

Then, by the projection formula, we have

$$j_{x!} \otimes j_{z,y*}\overline{\mathbb{Q}}_\ell[\ell(y)] = j_{x!}\Big(\overline{\mathbb{Q}}_\ell[\ell(x)] \otimes j_x^* j_{z,y*}\overline{\mathbb{Q}}_\ell[\ell(y)]\Big) = j_{x!} j_x^* j_{z,y*}\overline{\mathbb{Q}}_\ell[\ell(x) + \ell(y)]$$

and hence a corresponding formula for compactly supported cohomology

$$R\Gamma_c\Big(m^{-1}(zI), \big(j_{x!} \boxtimes_I j_{y*}\big)|_{m^{-1}(zI)}\Big) = R\Gamma_c\Big(\mathrm{Fl}_x, j_x^* j_{z,y*}\overline{\mathbb{Q}}_\ell\Big)[\ell(x) + \ell(y)]$$

Now, substituting $x = w$, $y = w^{-1}$, and $z = e$, we have $X = \mathrm{Fl}_w$ and hence

$$\big(j_{w!} *_I j_{w^{-1}*}\big)_I = R\Gamma_c\Big(\mathrm{Fl}_w, \overline{\mathbb{Q}}_\ell\Big)[2\ell(w)] = R\Gamma_c\Big(\mathbb{A}^{\ell(w)}, \overline{\mathbb{Q}}_\ell\Big)[2\ell(w)] = \overline{\mathbb{Q}}_\ell$$

Then, for $w = x = y \in S$, note that $j_{w!} *_I j_{w*}$ has support in $\mathrm{Fl}_s \coprod \mathrm{Fl}_e = \overline{\mathrm{Fl}}_s = \mathbb{P}^1$, by the property of proper pushforwards. Thus, it suffices to prove that $j_{w!} *_I j_{w*}$ has no support on wI. Note $j_w : X \hookrightarrow \mathrm{Fl}_w$ can be identified as $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, removing the

0 (or ∞, if we first embed $\mathrm{Fl}_w \hookrightarrow \mathbb{P}^1$ avoiding the 0). Hence, it suffices to prove that $R\Gamma_c\left(\mathbb{A}^1, j_*\left(\overline{\mathbb{Q}}_\ell\right)_{\mathbb{A}^1-0}\right) = 0$, which is well-known.

Finally, for $\ell(w) = 0$, $j_{w!} = j_{w*} = \delta_w$ is again the skyscraper at wI, so obviously the assertion holds. Thus we have proven (3).

$\square$

Corollary 2.1.6. *The above lemma holds with $j_{w!}$ replaced by $j^m_{w!}$, and j_{w*} replaced by j^m_{w*}.*

Proof. By doing the same process on the finite level $(\mathrm{Fl}_w)_0$ gives the result. $\square$

Lemma 2.1.7.

(1) For $w \in W^{\mathrm{aff}}$, we have non-trivial morphisms

$$\delta_e \to j_{w!}, j_{w*} \to \delta_e$$

whose (co)kernel does not contain δ_e in its Jordan-Hölder series.

(2) If $w = w_1 w_2 \in W^{\mathrm{ext}}$, $w_2 \in W^{\mathrm{aff}}$, and $\ell(w) = \ell(w_1) + \ell(w_2)$, then

$$\dim \mathrm{Hom}\left(j_{w_1!}, j_{w!}\right) = 1 = \dim \mathrm{Hom}\left(j_{w*}, j_{w_1*}\right)$$

and the unique non-zero map is injective (surjective).

Proof. We only prove the statements for $j_{w!}$, and the other side is obtained by Verdier duality.

(1) For $w \in W^{\mathrm{aff}}$ of length 1 (i.e. a simple reflection), note $\overline{\mathrm{Fl}}_w = \mathbb{P}^1$, so we have an exact sequence

$$0 \to \delta_e \to j_{w!} \to L_w \to 0$$

Then for general $u \in W^{\mathrm{aff}}$, we proceed by induction on $\ell(u)$. Suppose w is a simple reflection such that $\ell(uw)$, taking convolution $j_{u!} *_I (-)$ gives a triangle

$$j_{u!} \to j_{uw!} \to j_{u!} *_I L_w$$

We first prove the final term is a perverse sheaf. We let $I_w = I \sqcup IwI$ denote the parahoric subgroup of LG. Then, since $\overline{\mathrm{Fl}}_w = \mathbb{P}^1$, we have $\pi : LG/I \twoheadrightarrow LG/I_w$ a $\mathbb{P}^1$-fibration. Notice that on $\mathbb{P}^1$, we have $j_{!*}\left(\overline{\mathbb{Q}}_\ell[1]\right) = \overline{\mathbb{Q}}_\ell[1]$, where $j : \mathbb{A}^1 \to \mathbb{P}^1$ is the inclusion. Thus, we may compute $j_{u!} *_I L_w = \pi^*\pi_*(j_{u!})[1]$. Then, since $\pi \circ j_u$ is a locally closed affine embedding, we have $\pi^*\pi_*(j_{u!})[1]$ a perverse sheaf. Hence, the above triangle is an exact sequence of perverse sheaves. Now, since $j_{u!}$ only contains one δ_e, it suffices to show that $j_{u!} *_I L_w$ contains no δ_e. This is because the subquotients of $\pi^*\pi_*(j_{u!})[1]$ is of the form $\pi^*(L[1])$ (by adjunction, since π is proper, we have π^* both left and right adjoint to π_*), hence contains no δ_e.

(2) we have

$$\mathrm{Hom}\big(j_{w_1!}, j_{w!}\big) = \mathrm{Hom}\big(j_{w_1!}, j_{w_1!} *_I j_{w_2!}\big) = \mathrm{Hom}\big(\delta_e, j_{w_2!}\big)$$

since $j_{w_1^{-1}*} *_I j_{w_1!} = \delta_e$. So (2) follows from (1).

□

With these preparations, we may define the most important sheaves in $\mathbf{Perv}_I(\mathrm{Fl})$ we are considering in this paper.

Construction 2.1.8. The functor $\lambda \mapsto j_{\lambda*}$ for $\lambda \in X_*^+(T)$ actually extends to $\lambda \in X_*(T)$. Indeed, for $\lambda \in X_*(T)$ (or $X^*(T^\vee)$), we let $\lambda = \mu - \rho$ for $\mu, \nu \in X_*^+(T)$. Then, we define $J_\lambda = j_{t_\mu *} *_I j_{t_{-\nu}!}$. These objects are called Wakimoto sheaves. By definition, we have $J_{\lambda_1+\lambda_2} = J_{\lambda_1} *_I J_{\lambda_2}$ by the previous lemma on the computations of convolutions of j's.

Lemma 2.1.9. *We have* $\mathrm{RHom}\big(J_\lambda, J_\mu\big) = 0$ *unless* $\lambda \leq \mu$*, and when* $\lambda = \mu$*,* $\mathrm{RHom}\big(J_\mu, J_\mu\big) = \overline{\mathbb{Q}}_\ell$.

Proof. By Lemma 2.1.5, convolution against J_ν is invertible. So, we pick $\nu \gg 0$ such that $\nu + \lambda$ and $\nu + \mu$ are dominant. Hence we have

$$\mathrm{RHom}\big(J_\lambda, J_\mu\big) = \mathrm{RHom}\big(J_{\nu+\lambda}, J_{\nu+\mu}\big) = \mathrm{RHom}\big(j_{\nu+\lambda,*}, j_{\nu+\mu,*}\big)$$

and the computation of the final Ext groups are well-known. □

With some further work, we may compute the cohomology of Wakimoto sheaves through a similar process.

Proposition 2.1.10 (Lemma 4.5.1[7]). *For any* λ *we have* $H^n(\mathrm{Fl}, J_\lambda) = \overline{\mathbb{Q}}_\ell$ *if* $n = \langle \lambda, 2\rho \rangle$ *and 0 otherwise. In particular, for* λ *a coroot, we have the cohomology of* J_λ *concentrated in even degrees.*

Construction 2.1.11. Moreover, we may define the Wakimoto sheaves not just for $\lambda \in X_*(T)$, but for general $w \in W^{\mathrm{ext}}$. Let $w = t_\lambda \cdot w_f$, we may define $J_w = J_\lambda *_I j_{w_f *}$ (sorry about changing right $T_*(X)$ cosets to left cosets here, but we will not use the generalized Wakimoto sheaves too much).

Lemma 2.1.12. *The Wakimoto sheaves J_w for $w \in W^{\mathrm{ext}}$ are perverse.*

Proof. We first take $w = t_\lambda^{-1} t_\mu w_f$, where $\lambda, \mu \in X_*^+(T)$, and $w_f \in W_f$. Thus, we have

$$J_w = j_{t_{-\lambda}!} *_I j_{t_\mu w_f *}$$

Then, unwrapping the definition of the convolution map, we have

$$J_w = m_!\Big(j_{t_{-\lambda}!} \boxtimes_I j_{t_\mu w_f,*}\Big) = m_!\big(j_{t_{-\lambda}} \times^I \mathrm{id}_{\mathrm{Fl}}\big)_!\Big(\overline{\mathbb{Q}}_\ell[w(-\lambda)] \boxtimes_I j_{t_\mu w_f,*}\Big)$$

But notice that the composition of first two morphism is exactly the restricted multiplication map $m : \mathrm{Fl}_{-\lambda} \times^I \mathrm{Fl} \to \mathrm{Fl}$, which is an affine morphism. But notice that $f_!$ for an affine morphism is perverse left t-exact, so $J_w \in {}^pD^{\geq 0}(\mathrm{Fl})$. Dually, by writing

$$J_w = m_*\Big(\mathrm{id}_{\mathrm{Fl}} \times^I j_{t_\mu w_f}\Big)_*\Big(j_{t_{-\lambda},!} \boxtimes_I \overline{\mathbb{Q}}_\ell\big[\ell(t_\mu w_f)\big]\Big)$$

we have f_* for an affine morphism perverse right t-exact, hence combining with the above fact we have J_w a perverse sheaf. $\square$

For the general Wakimoto sheaves J_w, we have the following lemma.

Lemma 2.1.13. *We have $J_\lambda *_I J_w = J_{t_\lambda w}$ for $\lambda \in X_*(T)$ and $w \in W_f$.*

Proof. This follows from the definition of the generalized Wakimoto sheaves. $\square$

2.2. Gaitsgory's central functor

We now give a review of Gaitsgory's paper[10]. In this section, we set k to be either $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$, and all schemes considered will be over this k. We will construct a central functor

$$Z : \mathbf{Perv}_{L^+G}(\mathrm{Gr}) \to \mathbf{Perv}_I(\mathrm{Fl})$$

along with the following properties:

Theorem 2.2.1 (Theorem 1[10]). *There exists a functor*

$$Z : \mathbf{Perv}_{L^+G}(\mathrm{Gr}) \to \mathbf{Perv}_I(\mathrm{Fl})$$

with the following properties:

(1) For any $\mathcal{F} \in \mathbf{Perv}_{L^+G}(\mathrm{Gr})$, $Z(\mathcal{F})$ is convolution exact (see Definition 2.3.5).

(2) The image of Z is central, that is, there is a natural isomorphism of functors

$$Z(-) *_I (-) \simeq (-) *_I Z(-) : \mathbf{Perv}_{L^+G}(\mathrm{Gr}) \otimes \mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_I(\mathrm{Fl})$$

(3) Z is a monoidal functor between the convolution monoidal structures.

(4) The projection $\pi_! : \mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_{L^+G}(\mathrm{Gr})$ is a right inverse for Z.

Notice that the properties of Z for $k = \overline{\mathbb{F}}_q$ gives the properties for ℓ-adic sheaves, and $k = \mathbb{F}_q$ gives the properties for mixed sheaves.

Theorem 2.2.2 (Theorem 2[10]). *The functor Z is equipped with a nilpotent monodromy endomorphism*

$$M : Z(-) \to Z(-)(-1) \text{ or } M : Z(-) \to Z(-)$$

(the first endomorphism is for $k = \mathbb{F}_q$, where (-1) denotes the Tate twist in the mixed case, and the second endomorphism is for $k = \overline{\mathbb{F}}_q$) which is a tensor derivation of Z. That is, the following diagram commutes

$$\begin{tikzcd}
Z(\mathcal{F}_1) *_I Z(\mathcal{F}_2) \arrow[r, "M_{\mathcal{F}_1} * \mathrm{id} + \mathrm{id} * M_{\mathcal{F}_2}"] \arrow[d, "\simeq"] & Z(\mathcal{F}_1) *_I Z(\mathcal{F}_2)(-1) \arrow[d, "\simeq"] \\
Z(\mathcal{F}_1 *_{L^+G} \mathcal{F}_2) \arrow[r, "M_{\mathcal{F}_1 * \mathcal{F}_2}"] & Z(\mathcal{F}_1 *_{L^+G} \mathcal{F}_2)(-1)
\end{tikzcd}$$

The likewise square commutes for $k = \overline{\mathbb{F}}_q$, with the Tate twists removed.

These functors will be crucial to constructing the equivalence. We will construct this functor via nearby cycles, and prove centrality, which is analogous to the braiding of the category $\mathbf{Perv}_{L^+G}(\mathrm{Gr})$, via fusion. For these purposes, we fix a smooth curve X/k (not necessarily proper) in this subsection. We first recall the definition of the Beilinson-Drinfeld Grassmannian.

Definition 2.2.3. We define Gr_X to be the functor whose S-points are given by the set of triples $(y, \mathcal{F}_G, \beta)$, where $y \in X$ is an S-point, $\mathcal{F}_G$ a G-bundle over $X \times S$, and β is a trivialization of $\mathcal{F}_G$ outside the graph of $y : S \to X$.

We slightly modify the above definition to obtain a factorizable version of the flag variety.

Definition 2.2.4. We fix a point $x \in X$. We define Fl_X to be the ind-scheme whose S -points are given by quadruples $(y, \mathcal{F}_G, \beta, \varepsilon)$, where $(y, \mathcal{F}, \beta)$ are as above, and ε is a reduction of $\mathcal{F}_G|_{x \times S}$ to a B-bundle.

Proposition 2.2.5. *We have a projection* $\mathrm{Fl}_X \twoheadrightarrow \mathrm{Gr}_X$. *Moreover,* $\mathrm{Fl}_{X-x} \simeq \mathrm{Gr}_{X-x} \times G/B$, *and* $\mathrm{Fl}_x = \mathrm{Fl}$.

Proof. This is straightforward from the above moduli description. □

Construction 2.2.6. We first construct the nearby cycle functor used in this section. We fix $x \in X$ a k-point. We let $\mathcal{O}$ denote the Henselization of the local ring $\mathcal{O}_{X,x}$. Then, let the geometric generic point and the geometric special point of $\mathcal{O}$ be given by $\overline{\eta}$ and $\overline{s}$ respectively. We let $\overline{\mathcal{O}}$ denote the integral closure of $\mathcal{O}$ in the algebraic closure of its fraction field $\overline{K}$.

Then, for a scheme $Y \to X$, consider the following pullback diagram

$$\begin{tikzcd} Y_{\overline{s}} \arrow[r, "\overline{\imath}"] \arrow[d] & Y_{\overline{\mathcal{O}}} \arrow[d] & Y_{\overline{\eta}} \arrow[l, "\overline{\jmath}"'] \arrow[d] \\ \overline{s} \arrow[r, "i"] & \operatorname{Spec}\overline{\mathcal{O}} & \overline{\eta} \arrow[l, "j"'] \end{tikzcd}$$

We now construct the nearby cycles functor $\Psi : D(Y_{X-x}) \to D(Y_x)$ for $k = \mathbb{F}_q$. For a sheaf $\mathcal{F} \in D(Y_{X-x})$, we define

$$\Psi'(\mathcal{F}) = \overline{\imath}^* \overline{\jmath}_* \left(\mathcal{F}|_{Y_{\overline{\eta}}}\right)[-1] \in D(Y_{\overline{s}})$$

Note $Y_{\overline{s}} \to Y_s$ is a Galois covering with Galois group $\mathrm{Gal}\left(\overline{k}/k\right)$, so we need a final action of $\mathrm{Gal}\left(\overline{k}/k\right)$ on $\Psi'(\mathcal{F})$ to descend it to a sheaf in $D(Y_s) = D(Y_x)$. For this, notice that the Galois group $\mathrm{Gal}\left(\overline{K}/K\right)$ acts on $\overline{\eta}$, so acts on $\mathcal{F}|_{Y_{\overline{\eta}}}$, and hence on $\Psi'(\mathcal{F})$. Also, we have an exact sequence

$$1 \to I \to \mathrm{Gal}\left(\overline{K}/K\right) \to \mathrm{Gal}\left(\overline{k}/k\right) \to 1$$

we thus fix a splitting of this exact sequence $\iota : \mathrm{Gal}\left(\overline{k}/k\right) \to \mathrm{Gal}\left(\overline{K}/K\right)$, and thus define

$$\Psi(\mathcal{F}) = \left(\Psi'(\mathcal{F}), \iota\left(\mathrm{Gal}\left(\overline{k}/k\right)\right)\right) \in D(Y_x)$$

where the pair is the descent datum of the Galois action.

Moreover, there is an extra inertia action $I \circlearrowright \Psi(\mathcal{F})$. We define the unipotent nearby cycles $\Psi^{\mathrm{unip}}(\mathcal{F})$ to be the subsheaf of $\Psi(\mathcal{F})$ where I acts unipotently (recall a priori a finite index subgroup of I acts unipotently on $\Psi(\mathcal{F})$ by the Grothendieck's monodromy theorem). We only consider the pro-ℓ part of the tame inertia, since we are working with ℓ-adic sheaves. So we have a unipotent action $\mathbb{Z}_\ell(1) \circlearrowleft \Psi^{\text{unip}}(\mathcal{F})$. Then, we may take its ℓ-adic logarithm and obtain an $\mathbb{Z}_\ell$-module morphism

$$\mathbb{Z}_\ell(1) \to \operatorname{End}(\Psi^{\text{unip}}(\mathcal{F}))$$

By tensor-Hom adjunction, we have thus constructed a monodromy endomorphism

$$M : \Psi^{\text{unip}}(\mathcal{F}) \to \Psi^{\text{unip}}(\mathcal{F})(-1)$$

The most important aspect of this functor is that it preserves perverse sheaves (see Section 4.4[11]). That is, we have Ψ^{unip} restricts to a functor

$$\Psi^{\text{unip}} : \mathbf{Perv}(Y_{X-x}) \to \mathbf{Perv}(Y_x)$$

Since we will not use the non-unipotent version later, we will not use Ψ, and so we replace Ψ^{unip} by Ψ. Moreover, in the case we are concerned with, we will have $\Psi = \Psi^{\text{unip}}$.

Notice that in particular, the nearby cycles construction preserves the category of mixed sheaves by the same paper[11]. Thus, as a result, the later constructed functor Z also preserves mixed sheaves.

In the case of $k = \overline{\mathbb{F}}_q$, the $\mathbb{Z}_\ell$ still exists, but since the Galois group $\operatorname{Gal}\left(\overline{k}/k\right)$ is trivial, there is no data of the Tate twist. In this case, we obtain a plain nilpotent endomorphism $Z(-) \to Z(-)$.

Construction 2.2.7. We then consider some factorization constructions. We let Aut denote the Virasoro group, which is the ind-group scheme of automorphisms of $k[[t]]$. Thus by definition, Aut acts on L^+G, LG, and hence on Gr.

By definition, we have

$$\operatorname{Aut}(k) = \{a_0 + a_1 t + \cdots \in k[[t]] \mid a_0 \, \text{nilpotent}, a_1 \text{ invertible}\}$$

with multiplication given by composition. We define Aut_0 to be its subgroup of series with $a_0 = 0$.

For any curve X, we may construct an Aut_0-torsor over X called the coordinate scheme. The scheme Coord_X has points given by the moduli of (x, t_x), where $x \in X$, and $t_x \in \mathcal{O}^\wedge_{X,x}$ is a topological generator. This space moreover admits a Aut-action by noticing $\mathrm{Coord}_X|_{\mathbb{D}_x} = \mathrm{Aut}$ as a torsor, so gluing this construction together gives an Aut -action on Coord_X.

By definition, we have $\mathrm{Gr}_X = \mathrm{Coord}_X \times^{\mathrm{Aut}} \mathrm{Gr}$.

Construction 2.2.8. Finally we may construct the central functor. We first consider the composition of functors

$$Z : \mathbf{Perv}_{\mathrm{Aut}}(\mathrm{Gr}) \overset{\mathcal{O}*_{\mathrm{Aut}}(-)}{\longrightarrow} \mathbf{Perv}(\mathrm{Gr}_{X-x}) \overset{(-)\boxtimes\delta_{e_{G/B}}}{\longrightarrow} \mathbf{Perv}(\mathrm{Fl}_{X-x}) \overset{\Psi_X}{\longrightarrow} \mathbf{Perv}(\mathrm{Fl})$$

We then show it upgrades to a functor $\mathbf{Perv}_{L^+G}(\mathrm{Gr}) \to \mathbf{Perv}_I(\mathrm{Fl})$. First, notice any L^+G-equivariant sheaf in Gr is naturally Aut-equivariant since $\mathbf{Perv}_{L^+G}(\mathrm{Gr})$ is semisimple, and any simple sheaf is by definition Aut-equivariant. Then, we take an equivariant perverse sheaf $\mathcal{F} \in \mathbf{Perv}_{L^+G}(\mathrm{Gr})$. We consider $\mathrm{Supp}\, Z(\mathcal{F})$, and note I must act through a quotient I_k (here I_k denotes the preimage of B in L^kG rather than L^+G) on $\mathrm{Supp}\, Z(\mathcal{F})$, so it suffices to prove that $Z(\mathcal{F})$ is I_k-equivariant.

To prove this, we first recall that I is the pullback group

$$\begin{array}{ccc} I & \longrightarrow & B \\ \downarrow & & \downarrow \\ L^+G & \longrightarrow & G \end{array}$$

Since we may identify $L^+G = \mathrm{Hom}(k[[t]], G)$ with the Hom stack, we may define I^{glob} to be the group functor sending a scheme S to the maps from the localization of $X \times S$ around $x \times S$ to G, such that the restriction to $x \times S$ is mapped to B. This can be regarded as the "global" version of I, and admits a map $I^{\mathrm{glob}} \to I$ given by taking the formal power expansion at $x \times S$. Clearly, $I^{\mathrm{glob}} \to I \twoheadrightarrow I_k$ is a surjection. Thus, it suffices to prove I^{glob}-equivariance. However, I^{glob} admits the above global definition, so we take an open set $U \subset X \times S$ containing $x \times S$, and consider a map $U \to G$ restricting to a map $x \times S \to B$. By the compatibility of nearby cycles with smooth pullbacks, it suffices to prove for the following maps $\varphi_1, \varphi_2 : U \times_X \mathrm{Fl}_X \to \mathrm{Fl}_X$,

$$\Psi_{U \times_X \mathrm{Fl}_X}\left(\varphi_1^*\left(\mathcal{F}_{X-x} \boxtimes \delta_{e_{G/B}}\right)\right) \simeq \Psi_{U \times_X \mathrm{Fl}_X}\left(\varphi_2^*\left(\mathcal{F}_{X-x} \boxtimes \delta_{e_{G/B}}\right)\right)$$

where φ_1 is the projection, and φ_2, informally, is the action of I^{glob} through U on Fl_X. The verification of the above equivalence is done by the L^+G-equivariance. Readers interested in the details can refer to Proposition 4 of the original paper[10].

Definition 2.2.9. We now construct the global action flag variety that allows us to prove convolution exactness and centrality. We define $\mathrm{Fl}_X^{\text{act}}$ to be the functor sending S to quadruples $(y, \mathcal{F}_G, \beta', \varepsilon)$, where $(y, \mathcal{F}_G, \varepsilon)$ is as in the definition in Fl_X, and β' now is a trivialization of $\mathcal{F}_G$ outside $\Gamma_y \cup (x \times S)$.

Proposition 2.2.10. *We have* $\mathrm{Fl}_{X-x}^{\text{act}} \simeq \mathrm{Gr}_{X-x} \times \mathrm{Fl}$, *and* $\mathrm{Fl}_x^{\text{act}} \simeq \mathrm{Fl}$.

Proof. Likewise to the proof of the previous proposition. □

Construction 2.2.11. We now construct the central isomorphism. We define a functor

$$\mathcal{C} : \mathbf{Perv}_{\mathrm{Aut}}(\mathrm{Gr}) \times \mathbf{Perv}(\mathrm{Fl}) \to \mathbf{Perv}(\mathrm{Fl})$$

given by the composition

$$\mathbf{Perv}_{\mathrm{Aut}}(\mathrm{Gr}) \times \mathbf{Perv}(\mathrm{Fl}) \overset{(-)_{X-x} \boxtimes (-)}{\longrightarrow} \mathbf{Perv}(\mathrm{Fl}_{X-x}^{\text{act}}) \overset{\Psi_{\mathrm{Fl}^{\text{act}}}}{\longrightarrow} \mathbf{Perv}(\mathrm{Fl})$$

Then, Gaitsgory constructed isomorphisms

$$Z(\mathcal{F}) *_I \mathcal{G} \simeq \mathcal{C}(\mathcal{F}, \mathcal{G}) \simeq \mathcal{G} *_I Z(\mathcal{F})$$

which gives the first two desired properties (see the proof of Proposition 6[10]).

Definition 2.2.12. We finally construct the global convolution flag variety that allows us to prove the monoidality of the central functor. Recall we have

$$\mathrm{Fl}^{\text{conv}} = \mathrm{Fl} \times^I I \backslash \mathrm{Fl}$$

is used in the definition of the convolution product of two sheaves. So, we make the following definition. We let $\mathrm{Fl}_X^{\text{conv}}$ denote the functor sending S to the septuples $\left(y, \mathcal{F}_G, \tilde{\beta}, \varepsilon, \mathcal{F}_G^1, \beta^1, \varepsilon^1\right)$, where $y : S \to X$ is an S-point, $\mathcal{F}_G$ and $\mathcal{F}_G^1$ are G-torsors, β^1 a trivialization of $\mathcal{F}_G^1$ outside Γ_y, $\tilde{\beta}$ an isomorphism of $\mathcal{F}_G$ and $\mathcal{F}_G^1$ outside Γ_y, and finally ε and ε^1 are reductions of $\mathcal{F}_G$ and $\mathcal{F}_G^1$ to B at $x \times S$.

This time, we have identifications

$$\mathrm{Fl}_{X-x}^{\text{conv}} = \mathrm{Gr}_{X-x}^{\text{conv}} \times G/B \times G/B, \mathrm{Fl}_x^{\text{conv}} = \mathrm{Fl}^{\text{conv}}$$

Construction 2.2.13. For the monoidality of Z, it suffices to prove

$$\Psi_{\mathrm{Fl}_X^{\mathrm{conv}}}\left(\mathcal{F}^1_{X-x} \boxtimes \mathcal{F}^2_{X-x} \boxtimes \delta_{e_{G/B}} \boxtimes \delta_{e_{G/B}}\right) \simeq Z(\mathcal{F}^1)\, \tilde{\boxtimes}\, Z(\mathcal{F}^2) \in \mathbf{Perv}(\mathrm{Fl}^{\mathrm{conv}})$$

which is likewise to the above constructions. In particular, we will use that taking nearby cycles commutes with taking proper pushforwards.

For (4) of Theorem 2.2.1, this simply follows from the compatibility of nearby cycles with proper pushforwards, namely by applying this fact to the projection $\pi : \mathrm{Fl}_X \to \mathrm{Gr}_X$. In particular, since after pushing forward we are doing nearby cycles with respect to a trivial family, the functor clearly gives id. This functor preserves weights, so $\pi_!(Z(-)) = \mathrm{id}$ holds even in the mixed setting.

We have finished the review of the proof of Theorem 1 in the paper[10]. The fact that Z admits a tensor derivation M follows from the fact that the compatibility of nearby cycles with proper pushforwards respects the action of the inertia group.

2.3. The Wakimoto filtration

Now, we construct a filtration on the above central functor by Wakimoto sheaves. More explicitly, we have the following proposition.

Proposition 2.3.1. *We identify the categories $\mathbf{Perv}_{L^+G}(\mathrm{Gr}) \simeq \mathbf{Rep}\, G^\vee$. Also, we fix a total ordering $\overset{t}{\leq}$ on $X_*(T)$ compatible with the group structure and with the partial ordering $\leq$. Then, for any $V \in \mathbf{Rep}\, G^\vee$, the sheaf $Z(V)$ has a unique filtration indexed by $\left(X_*(T), \overset{t}{\leq}\right)$, such that the associated graded pieces satisfies*

$$\mathrm{gr}_\nu(Z(V)) := Z(V)_{\overset{t}{\leq}\nu} / Z(V)_{\overset{t}{<}\nu} = J_\nu \otimes W_V^\nu$$

where J_ν is the Wakimoto sheaves, and W_V^ν is some finite dimensional vector space.

Lemma 2.3.2. *Such a filtration is unique.*

Proof. By Lemma 2.1.9, since the Ext groups of Wakimoto sheaves $\mathrm{RHom}(J_\lambda, J_\mu)$ are non-zero only when $\lambda \leq \mu$, upon fixing the total ordering compatible with the partial ordering and fixing the graded pieces, one can show that there is only one way to put the graded pieces together, and hence the filtration is unique. $\square$

We now move on to prove the existence of the filtration.

Definition 2.3.3. For $X \in D_I(\mathrm{Fl})$, we set

$$W_X^* = \{w \in W^{\mathrm{ext}} | j_w^*(X) \neq 0\}$$

and

$$W_X^! = \left\{w \in W^{\mathrm{ext}} | j_w^!(X) \neq 0\right\}$$

Lemma 2.3.4. *For $X \in D_I$, there is a finite subset $S \subset W^{\mathrm{ext}}$, such that we have for all $w \in W^{\mathrm{ext}}$, we have*

$$W^!_{j_{w*} *_I X}, W^*_{j_{w!} *_I X} \subset w \cdot S$$

and

$$W^!_{X *_I j_{w*}}, W^*_{X *_I j_{w!}} \subset S \cdot w$$

Proof. By proper base change, the support of $j_{w!} *_I X$ is contained in the convolution of Fl_w and $\mathrm{Supp}\, X$. Hence to prove the assertion $W^*_{j_{w!} *_I X} \subset w \cdot S$, it suffices to prove the corresponding assertion for the convolution of sets. Since X is an I-equivariant sheaf, its support $\mathfrak{S}$ is an I-stable set. It suffices to show that there is some $S \subset W^{\mathrm{ext}}$, the convolution

$$\mathrm{Fl}_w *_I \mathfrak{S} \subset \bigsqcup_{v \in S \cdot w} \mathrm{Fl}_v$$

By decomposing $\mathfrak{S}$ into I-orbits, the assertion follows from induction on the length of elements. The other three assertion is similar. □

Definition 2.3.5. We define a perverse sheaf $\mathcal{F} \in \mathbf{Perv}_I(\mathrm{Fl})$ to be convolution exact if $\mathcal{F} *_I (-)$ is a functor from $\mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_I(\mathrm{Fl})$.

Proposition 2.3.6. *Any convolution exact object in $\mathbf{Perv}_I(\mathrm{Fl})$ has a Wakimoto filtration. If such object is central, only the Wakimoto sheaves J_λ for $\lambda \in X_*(T)$ will appear in the filtration.*

Proof. Let X be convolution exact, and let S be as in Lemma 2.3.4. Then, writing the set $S = \{s_i\} = \{\lambda_i w_i\}$ such that $\lambda_i \in X_*(T)$ and $w_i \in W_f$, we choose $\nu_0 \ll 0 \in X_*(T)$ such that $-(\nu_0 + \lambda_i)$ is strictly dominant (that is, dominant and not lying in any wall). Thus, by a direct calculation, we have $J_{t_{-\nu_0} s_i} = j_{t_{-\nu_0} s_i !}$ by expanding the definition of Wakimoto sheaves. By Lemma 2.1.12, we have

$$j_{t_{-\nu_0}!} *_I X = J_{-\nu_0} *_I X \in \mathbf{Perv}_I(\mathrm{Fl})$$

since X is convolution exact. We use that this sheaf lies in the category ${}^pD_I^{\geq 0}(\mathrm{Fl})$. Then, by the gluing property of the perverse t-structure along closed and its complement open subschemes, we have by Lemma 2.3.4

$$\mathrm{Supp}\Big(j_{t_{-\nu_0}!} *_I X\Big) \subset \bigsqcup_{s \in S} \mathrm{Fl}_{t_{-\nu_0} s}$$

so we have $J_{\nu_0} *_I X$ lie in the subcategory generated by $j_{t_{\nu_0} \cdot s,!}[i]$ for $i \geq 0$ and $s \in S$, since the sheaf is perverse and constructible with respect to this stratification. Then, by construction of ν_0, we have by definition $j_{t_{-\nu_0} \cdot s,!} = J_{t_{-\nu_0} \cdot s}$. To sum up, we have

$$J_{-\nu_0} *_I X \in \mathrm{Span}\Big\{J_{-\nu_0 \cdot s}[i] \mid i \geq 0, s \in S\Big\}$$

and hence by $J_\lambda *_I J_\mu = J_{\lambda+\mu}$,

$$J_\nu *_I X \in \mathrm{Span}\Big\{J_{t_\nu \cdot s}[i] \mid i \geq 0, s \in S\Big\}$$

for any ν. Then, choosing ν a dominant coweight such that $\nu + \lambda_i$ are all dominant, we have

$$J_\nu *_I X \in \mathrm{Span}\Big\{j_{t_\nu \cdot s,*} \mid s \in S\Big\}$$

where we do not need the shift because $J_\nu *_I X$ is already perverse. This proves the existence of the filtration.

For the second part specifying the graded pieces in the filtration, we choose λ a dominant coweight such that $\lambda \cdot S$ and $S \cdot \lambda$ are all in W_f-orbits of strictly dominant coweights. By centrality, we have

$$W^!_{X *_I J_\lambda} = W^!_{J_\lambda *_I X} \subset W_f \cdot X_*^{++}(T) \cap X_*^{++}(T) \cdot W_f = X_*^{++}(T)$$

hence the claim. $\square$

Now, by the results in the previous subsection, we have constructed the filtration.

2.4. The Iwahori–Whittaker sheaves

It turns out that we cannot directly prove the main equivalence. This is mainly because $D({}^f\mathcal{P}_I) \neq {}^fD_I(\mathrm{Fl})$. Instead, we will first prove the following

Theorem 2.4.1 (Theorem 3[6]). *We have an equivalence of categories*

$$\Phi_{\mathcal{IW}} : D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \to D_{\mathcal{IW}}(\mathrm{Fl})$$

where $D_{\mathcal{IW}}(\mathrm{Fl})$ *is a certain category of Iwahori–Whittaker sheaves that we will define below.*

and then relate the category of Iwahori–Whittaker sheaves to the original category.

Theorem 2.4.2 (Theorem 2[6]). *The averaging functor is an equivalence*

$$\mathrm{Av}_\Psi : {}^f\mathcal{P}_I \xrightarrow{\simeq} \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$$

Using the fact that $D^b(\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})) \simeq D_{\mathcal{IW}}(\mathrm{Fl})$ we will obtain the final result.

The way of passing through the Iwahori–Whittaker sheaves is partially inspired by the method of Whittaker functions in automorphic theory. Moreover, these sheaves enjoy the following property.

Proposition 2.4.3 (Lemma 1[6]). *By the formalism of highest weight categories, we have a natural equivalence*

$$D_{\mathcal{IW}}(\mathrm{Fl}) = D(\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl}))$$

and hence we have fully faithful maps

$$D_{\mathcal{IW}}(\mathrm{Fl}) \subset D(\mathrm{Fl}), \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl}) \subset \mathbf{Perv}(\mathrm{Fl})$$

Definition 2.4.4. Consider $K' \subset K$ algebraic groups acting on some X. Moreover suppose K/K' is of finite type. Then, we define functors on the category of equivariant sheaves (for any sheaf theory).

(1) Let $\mathrm{res}^K_{K'} : \mathrm{Sh}^K(X) \to \mathrm{Sh}^{K'}(X)$ denote the natural forgetful functor, or the $*$-pullback functor of sheaves along the projection $X/K' \to X/K$.
(2) Its left adjoint $\mathrm{coind}^K_{K'} : \mathrm{Sh}^{K'}(X) \to \mathrm{Sh}^K(X)$. This is the shifted !-pushforward functor along the projection (since π is smooth).
(3) Its right adjoint $\mathrm{ind}^K_{K'} : \mathrm{Sh}^{K'}(X) \to \mathrm{Sh}^K(X)$. This is the $*$-pushforward functor along the projection.

Construction 2.4.5. Let G be an algebraic group acting on X. Let $\mathcal{L}$ be a character sheaf over G, that is, we have a morphism $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$, here m stands for the multiplication map of the group $m : G \times G \to G$.

We define the category of $(G, \mathcal{L})$-equivariant sheaf over X to be defined via the following descent:

$$\mathrm{Sh}^{G,\mathcal{L}}(X) = \lim\Big(\mathrm{Sh}(X) \rightrightarrows \mathrm{Sh}(G \times X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Sh}(G \times G \times X)\cdots\Big)$$

where the morphisms in the limit diagram are given by either pulling back along the action map of G on X $a : G \times X \to X$, the multiplication map $m : G \times G \to G$, or by taking exterior tensor product with the character sheaf.

By definition, $\mathrm{Sh}^{G}(X) = \mathrm{Sh}^{(G,\mathcal{O})}(X)$.

Such construction is also compatible with the pro-group acting on ind-scheme formalism.

Construction 2.4.6. We consider the negative Iwahori I^{-}. We denote its pro-unipotent radical by I_0^{-}. Also, consider the loop group of the negative unipotent subgroup LN^{-}. Consider the following additive character on LN^{-}:

$$LN^{-} \to N^{-} \to N^{-}/[N^{-}, N^{-}] \xrightarrow{\Sigma} \mathbb{G}_a \to \mathbb{A}^1$$

where the first morphism is given by taking residues, the second is the natural projection, and the third is obtained by identifying $N^{-}/[N^{-}, N^{-}]$ with a direct product of $\mathbb{G}_a$'s indexed by the simple roots and taking sums. This character induces a character $\Psi : I_0 \to \mathbb{G}_a$ given by first restricting to the subgroup $I_0 \cap N$ and then extending to the group I_0.

Then, since we are in characteristic p, consider the Artin-Schreier covering

$$AS = x^p - x : \mathbb{A}^1 \twoheadrightarrow \mathbb{A}^1$$

We have

$$AS_*\big(\overline{\mathbb{Q}}_\ell\big) = \bigoplus_{\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times} \mathcal{L}_\psi$$

where $\mathcal{L}_\psi$ is a character sheaf over $\mathbb{G}_a = \mathbb{A}^1$ where the Galois group acts as ψ. We fix any ψ, and denote the corresponding $\mathcal{L}_\psi$ by $\mathbb{AS}$. Now, pulling this sheaf back along the map $LN^{-} \to \mathbb{G}_a$ gives a character sheaf over LN^{-}.

We let $D_{\mathcal{JW}}(\mathrm{Fl})$ and $\mathbf{Perv}_{\mathcal{JW}}(\mathrm{Fl})$ denote the corresponding $(I_0, \Psi^*(\mathbb{AS}))$-equivariant derived categories of ℓ-adic sheaves and perverse sheaves (using the pro-group acting on ind-schemes formalism as in the paper[9]). Since I_0 is again contractible, and the I_0^{-}-

orbits in Fl are affine, $D_{\mathcal{IW}}(\mathrm{Fl}) \subset D(\mathrm{Fl})$ is a full subcategory, and $D^b(\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})) = D_{\mathcal{IW}}(\mathrm{Fl})$.

Construction 2.4.7. We now construct certain Iwahori–Whittaker perverse sheaves. Let $i_w : \mathrm{Fl}^w \to \mathrm{Fl}$ denote the I_0^--orbit in Fl corresponding to $I_0^- wI$. We let $w \in {}^fW$ denote the minimal length element in the left coset $W_f w$. Then, for $w \in {}^fW$, we have $\mathrm{Stab}_{I_0^-}(wI) \subset \ker \Psi$. Hence, the definition $\Psi_w : \mathrm{Fl}^w \to \mathbb{G}_a$ can be obtained by factoring $\Psi : I_0^- \to \mathbb{G}_a$. Thus, we define for $w \in {}^fW$

$$\Delta_w = i_{w!}\Psi_w^*(\mathbb{AS})[\ell(w)], \nabla_w = i_{w*}\Psi_w^*(\mathbb{AS})[\ell(w)]$$

We sometimes abuse notation and define $\Delta_\lambda := \Delta_w$ and $\nabla_\lambda := \nabla_w$, where $\lambda \in X_*(T)$, and w is the element in W^{ext} such that $w \in W_f t_\lambda \subset W^{\mathrm{ext}}$ is of minimal weight.

Then, we define $\mathrm{IC}_w = i_{w!*}\Psi_w^*(\mathbb{AS})[\ell(w)]$. Since $\Psi_w^*(\mathbb{AS})$ is an irreducible local system, IC_w is a simple perverse sheaf viewed in $\mathbf{Perv}(\mathrm{Fl})$. Clearly, these simple objects generate $\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$.

Construction 2.4.8. We may relate the category of Iwahori–Whittaker sheaves and the I-equivariant sheaves on the flag variety through the following averaging functor. We define $\mathrm{Av}_\Psi : D_I(\mathrm{Fl}) \to D_{\mathcal{IW}}(\mathrm{Fl})$ by $\mathcal{F} \mapsto \Delta_e *_I \mathcal{F}$. Here Av stands for averaging. $\Delta_e *_I \mathcal{F}$ is indeed an Iwahori–Whittaker sheaf, since pullbacks commute with convolution, I_0^- is conjugate to I_0 by w_0, $\mathcal{F}$ is I_0-invariant, and Δ_e transforms by tensoring with the character sheaf.

We now present some basic properties of the functor Av_Ψ.

Proposition 2.4.9.

(1) $\mathrm{Av}_\Psi(L_w) \neq 0$ *iff* $w \in {}^fW$.

(2) $\Delta_e = \nabla_e$.

(3) For any $w \in W$*, we factor it as* $w_f \cdot w'$*, where* $w_f \in W_f$ *and* $w' \in {}^fW$*. Then, we have*

$$\mathrm{Av}_\Psi(j_{w!}) = \Delta_{w'}, \mathrm{Av}_\Psi(j_{w*}) = \nabla_{w'}$$

Proof.

(1) If $w \in {}^fW$, then the support of $\Delta_e \boxtimes_I L_w$ is $I_0^- I \times^I \overline{\mathrm{Fl}}_w$. Notice that $I_0^- I = I_0 w_0 I = I w_0 I$, so by the minimality of the length of w, we have the convolution map

$\mathrm{Fl} \times^I \mathrm{Fl} \to \mathrm{Fl}$ the identity on the generic point of the support of $\Delta_e \boxtimes_I L_w$. Thus, $\Delta_e *_I L_w \neq 0$. Then for the converse, if the length of w is not minimal in $W_f \cdot w$, say $sw \leq w$ for some simple reflection, then we have L_w equivariant with respect to some minimal parahoric I_s. Also, the functor $\Delta_e *_I (-) : D_{I_s}(\mathrm{Fl}) \to D_{\mathcal{IW}}(\mathrm{Fl})$ factors through the projection π_{s*}, where $\pi_s : LG/I \twoheadrightarrow LG/I_s$ is the projection. But notice $\pi_{s*}(\Delta_e) = 0$ since the character Ψ is non-trivial on the image of the Iw_0I, so we are done.

(2) Firstly, notice that Ψ is non-vanishing on any point of

$$\overline{\mathrm{Fl}}^e - \mathrm{Fl}^e = \bigcup_{w \leq w_0} IwI$$

Thus $\Delta_e = \nabla_e$ are both the constant sheaves supported on Fl^e.

(3) It suffices to prove $\mathrm{Av}_\Psi(j_{w!}) = \Delta_{w'}$, since the other one follows from duality. Firstly, if already we have $w \in {}^fW$, the assertion is trivial since analogous to (1), we have the comparison map an equivalence on Fl^w, and analogous to (2), we have LHS vanishes outside Fl^w (RHS obviously vanishes outside Fl^w). For general w, the formula follows from

$$\Delta_e *_I j_{w!} = \Delta_e *_I j_{w_f!} *_I j_{w'!} = \left(\Delta_e *_I j_{w_f!}\right) *_I j_{w'!} = \Delta_e *_I j_{w'!} = \Delta_{w'!}$$

where the first equality is by Lemma 2.1.5, the second by the associativity of convolution, and the third follows from Lemma 2.1.7.

$\square$

It turns out that Av_Ψ is an equivalence of categories ${}^f\mathcal{P} \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$. However, we will not be able to prove this for now. Instead, we will prove the following proposition and delay the proof of the above fact until the end of the proof of the main theorem Theorem 1.2.1.

Proposition 2.4.10.

(1) We have the functor $\mathrm{Av}_\Psi : D_I(\mathrm{Fl}) \to D_{\mathcal{IW}}(\mathrm{Fl})$ restricts to a functor $\mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$.

(2) This functor factors through the quotient $\mathbf{Perv}_I(\mathrm{Fl}) \twoheadrightarrow {}^f\mathcal{P}$.

(3) The induced functor ${}^f\mathcal{P} \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$ is fully faithful.

To prove this, we construct its right inverse.

Construction 2.4.11. We first wish to construct a functor $\mathrm{Av}_I : D_{\mathcal{JW}}(\mathrm{Fl}) \to D_I(\mathrm{Fl})$. However, since $D_{\mathcal{JW}}(\mathrm{Fl})$ is a twisted equivariant derived category, direct induction does not seem to work. In the setting here, we are lucky enough: we have $I_0^- \cap I = tL^+G$, and the residue map $LN^- \to N$ is obviously trivial on $LN^- \cap tL^+G$. Hence, the restriction to $I_0^- \cap I = tL^+G$ gives a non-twisted tL^+G-equivariant sheaf. We may define Av_I to be $\mathrm{ind}_{tL^+G}^{I} \mathrm{res}_{tL^+G}^{I_0^-}$.

Then, to obtain an inverse on the level of perverse sheaves, we consider the composition

$$\mathbf{Perv}_{\mathcal{JW}}(\mathrm{Fl}) \overset{\mathrm{Av}_I}{\longrightarrow} D_I(\mathrm{Fl}) \overset{{}^pH^{\ell(w_0)}}{\longrightarrow} \mathbf{Perv}_I(\mathrm{Fl}) \twoheadrightarrow {}^f\mathcal{P}$$

where w_0 is the longest element in the finite Weyl group. We denote this functor by F (since we will not really use this functor after the proof of Proposition 2.4.10).

Lemma 2.4.12. *We have* $F \circ \mathrm{Av}_\Psi = \mathrm{id}$ *restricted to* ${}^f\mathcal{P}$.

Proof. We wish to compute for $\mathcal{F} \in \mathbf{Perv}_I(\mathrm{Fl})$ the object $F \circ \mathrm{Av}_\Psi\big({}^f\mathcal{F}\big)$. Expanding definitions, we have

$$F \circ \mathrm{Av}_\Psi\big({}^f\mathcal{F}\big) = {}^f\big({}^pH^{\ell(w_0)}\big(\mathrm{Av}_I\big(\Delta_e *_I {}^f\mathcal{F}\big)\big)\big)$$

We break this series of functors from inside.

We have first

$$\mathrm{Av}_I\big(\Delta_e *_I {}^f\mathcal{F}\big) = {}^f(\mathrm{Av}_I(\Delta_e) *_I \mathcal{F})$$

this is because projecting to ${}^f\mathcal{P}$ commutes with everything, and left Iwahori averaging by construction commutes with convolution on the right.

Then, we must compute the cohomology

$${}^pH^{\ell(w_0)}\Big({}^f(\mathrm{Av}_I(\Delta_e) *_I \mathcal{F})\Big) = {}^f\big({}^pH^0(\mathrm{Av}_I(\Delta_e[\ell(w_0)]) *_I \mathcal{F})\big)$$

where we put the cohomology index inside, and move the projection outside. We first compute ${}^pH^0\Big(\mathrm{Av}_I\Big(\Delta_{e[\ell(w_0)]}\Big)\Big)$. We first construct a morphism $j_{w_0!} \to {}^pH^0\Big(\mathrm{Av}_I\Big(\Delta_{e[\ell(w_0)]}\Big)\Big)$ with cofiber lying in ${}^pD_I^{>0}$.

For this morphism, note

$$\mathrm{Hom}_{D_I(\mathrm{Fl})}(X, \mathrm{Av}_I(\Delta_e)[\ell(w_0)]) = \mathrm{Hom}_{D(\mathrm{Fl})}(X, \Delta_e[\ell(w_0)])$$

For the right hand side, note $D_{tL^+G}(\mathrm{Fl}) \subset D(\mathrm{Fl})$ is a full subcategory since tL^+G is unipotent (contractible), and the sheaves should lie in $D_{tL^+G}(\mathrm{Fl})$. Then, the proof of Proposition 2.4.9 gives $\mathrm{RHom}(L_w, \Delta_e) = 0$ for all $e \neq w \in W_f$, and $\mathrm{Hom}(\delta_e, \Delta_e[\ell(w_0)]) = \overline{\mathbb{Q}}_\ell$. Hence by Lemma 2.1.7, note that $j_{w!}$ (for $w \in W_f$) contains only $L_{w'}$ ($w' \in W_f$ also) in its composition series and exactly one δ_e, so we have

$$\mathrm{Hom}_{D_I}(j_{w!}, \mathrm{Av}_I(\Delta_e)[\ell(w_0)]) = \overline{\mathbb{Q}}_\ell$$

and for $w < w'$, the composition of the inclusion with the unique non-zero map

$$j_{w!} \hookrightarrow j_{w'!} \to \mathrm{Av}_I(\Delta_e)[\ell(w_0)]$$

is also non-zero again by Lemma 2.1.7. Hence, the cofiber C of $j_{w_0!} \to \mathrm{Av}_I(\Delta_e)[\ell(w_0)]$ has the property that for every $w \in W_f$, $\mathrm{Hom}(j_{w!}, C[i]) = 0$ for $i \leq 0$. Moreover, since $\mathrm{Av}_I(\Delta_e)$ is supported on the finite part IW_fI, we have $C \in {}^pD^{>0}$. Moreover, the map $j_{w_0!} \to {}^pH^0(\mathrm{Av}_I(\Delta_e)[\ell(w_0)])$ is an isomorphism in the category $\mathbf{Perv}_I(\mathrm{Fl})$ since its kernel must be of the form $j_{w!}$ for some $w \in W_f$, which is not possible since

$$j_{w!} \hookrightarrow j_{w_0!} \to \mathrm{Av}_I(\Delta_e)[\ell(w_0)]$$

is non-zero.

Hence, we have finished the following reduction.

$${}^f({}^pH^0(\mathrm{Av}_I(\Delta_e[\ell(w_0)]) *_I \mathcal{F})) = {}^f\left({}^pH^0\left(j_{w_0!} *_I \mathcal{F}\right)\right)$$

Finally, since ${}^fL_w = 0$ for all $e \neq w \in W_f$, we have the final perverse sheaf ${}^f\mathcal{F}$ itself.

Hence F is indeed a right inverse! □

Proof. We now prove Proposition 2.4.10.

(1) To prove that Av_Ψ restricts to a functor of perverse sheaves, we need the following characterization (by the gluing definition of perverse t-structures) of their perverse t-heart. Namely, we have

$$\mathbf{Perv}_I(\mathrm{Fl}) = \mathrm{Span}\{j_{w!}[i], i \geq 0\} \cap \mathrm{Span}\{j_{w*}[i], i \leq 0\}$$

and

$$\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl}) = \mathrm{Span}\{\Delta_{w!}[i], i \geq 0\} \cap \mathrm{Span}\{\nabla_{w*}[i], i \leq 0\}$$

However, we know by Proposition 2.4.9(3) that Av_Ψ preserves exactly these objects. Hence Av_Ψ restricts to a functor of the categories of perverse sheaves.

(2) This is by the universal property of Serre quotients and Proposition 2.4.9(1).
(3) With the previous lemma, we are very close to full faithfulness. For this, we need the following claim on Abelian categories: if a functor $F : \mathcal{A} \to \mathcal{B}$ of Abelian categories satisfies the following list of properties
   (a) F is exact.
   (b) Every object of $\mathcal{A}$ is of finite length.
   (c) F induces an isomorphism on the Hom groups

$$\mathrm{Hom}_{\mathcal{A}}(L_1, L_2) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{B}}(F(L_1), F(L_2))$$

   for all irreducible objects $L_i \in \mathcal{A}$.
   (d) F has a right inverse.

   then it is fully faithful. This is a simple fact obtained by applying induction on the length of $A, B \in \mathcal{A}$ to prove $\mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$.

   Now, we apply this claim to the functor Av_{Ψ}. Note that (1) provides that Av_{Ψ} is t-exact, so we have Av_{Ψ} exact after restricting to the category of perverse sheaves. (2) is naturally satisfied by the category of perverse sheaves. Then, note

$$\mathrm{Av}_{\Psi}(L_w) = \mathrm{Av}_{\Psi}(\mathrm{im}(j_{w!} \to j_{w*})) = \mathrm{im}(\Delta_w \to \nabla_w)$$

   are simple, and these objects are distinct since they have different supports. Thus by the property of morphism of simple objects, (3) is satisfied. (4) is verified in the previous lemma, so we are done.

□

2.5. Hecke algebras

In this subsection, we identify the K^0-groups of certain sheaf categories, allowing us to explicitly handle the computations of some Euler characteristics, for example in Lemma 3.3.5.

Definition 2.5.1. Recall that for a reductive group G, we define its affine Hecke algebra $\mathcal{H}$ to be generated by the following objects and relations:
(1) Let $\mathcal{H}$ be a $\mathbb{Z}[v, v^{-1}]$-algebra with generators H_w for $w \in W^{\mathrm{ext}}$.
(2) The relations are given by $(H_s + v)(H_s - v^{-1}) = 0$ for $s \in S$ a simple reflection, and $H_u H_v = H_{uv}$ for $\ell(u) + \ell(v) = \ell(uv)$.

We let $\mathcal{M}^{\text{sph}} = \mathcal{H} \otimes_{\mathcal{H}_f} \text{triv}$, where $\mathcal{H}_f$ is the finite Weyl group.

Recall that the Hecke algebra admits a Kazhdan-Lusztig basis B_w. Clearly, we have $\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} = \mathbb{Z}[W]$. We will also denote the basis of this algebra by H_w.

Proposition 2.5.2. *We have $\mathbb{Z}[W] = K^0(D_I(\text{Fl}))$, where the morphism is given by $(-1)^{\ell(w)} H_w \mapsto [j_{w*}]$. Moreover, $(-1)^{\ell(w)} B_w \mapsto [L_w]$.*

Proof. First, note j_{w*} generates the category $D_I(\text{Fl})$, so this morphism is surjective. Then, since these sheaves have different supports, taking the homology of the stalks gives a function

$$\varphi_w : K^0(D_I(\text{Fl})) \to \mathbb{Z}, \varphi_w([\mathcal{F}]) = \sum_{n \in \mathbb{Z}} (-1)^n \dim H^n(\text{Fl}_w, j_w^* \mathcal{F})$$

satisfies $\varphi_w([j_{u*}]) = (-1)^{\ell(w)} \delta_{w,u}$, so these $[j_{w*}]$ are all linearly independent. Hence, the map is injective.

Thus, the map is a bijection of Abelian groups, and it now suffices to show that it is a morphism of rings. But this is equivalent to

$$[j_{u*} *_I j_{v*}] = [j_{u*}][j_{v*}]$$

This is true because for $\ell(v) = 0$, or $\ell(uv) = \ell(u) + \ell(v)$, the statement for underlying sheaf holds by Lemma 2.1.5. If $\ell(v) = 1$, $\ell(uv) = \ell(u) - 1$, so since v is a simple reflection, we have

$$[j_{uv*}][j_{v*}] = [j_{u*}]$$

and since the simple perverse sheaves are preserved under Verdier duality, we have, by explicitly writing out the exact sequence on $\mathbb{P}^1 \simeq \text{Fl}_w$,

$$[j_{v*}] = [j_{v!}]$$

Hence,

$$[j_{uv*}] = [j_{uv*}][j_{v*}][j_{v!}] = [j_{u*}][j_{v!}] = [j_{u*}][j_{v*}]$$

The part on Kazhdan-Lusztig basis is standard. $\square$

Then, we relate this construction to the K-group of Affine grassmannians.

Proposition 2.5.3. *We have an isomorphism*

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\text{sph}} = K^0(D_I(\text{Gr}))$$

such that the projection $\mathcal{H} \twoheadrightarrow \mathcal{M}^{\mathrm{sph}}$ *corresponds to* $\pi_* : \mathrm{Fl} \twoheadrightarrow \mathrm{Gr}$.

Proof. Likewise as in the above. □

Definition 2.5.4. We define $\theta_\lambda = [J_\lambda] \in H[W]$ to be the class representing the Wakimoto sheaves. We may upgrade this construction to the Hecke algebra by taking $\theta_\lambda = H_\mu H_\nu^{-1}$ for $\mu - \nu = \lambda$ and $\mu, \nu \in X_*(T)^+$.

Proposition 2.5.5. *For any* $V \in \mathbf{Rep}\, G^\vee$, *we have*

$$[Z(V)] = \sum_{\lambda \in X_*(T)} \dim V^{T^\vee = \lambda} \cdot \theta_\lambda \in K^0(D_I(\mathrm{Fl}))$$

Proof. The proof of this proposition relies on Proposition 3.2.9. The rest follows from the Wakimoto filtration. □

Then, to give a more delicate characterization of the nilpotent endomorphism M_V, we give another calculation (though standard) on the Hecke algebras and K-groups.

Proposition 2.5.6. *We have an isomorphism*

$$R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} = K^0\big(D_{I,\mathrm{mixed}}(\mathrm{Fl})\big)$$

sending $1 \otimes H_w \mapsto (-1)^{\ell(w)}[j^m_{w*}]$ *and* $v \mapsto \left[\overline{\mathbb{Q}}_\ell(-1/2)\right]$. *Here* $R = K^0\big(D_{\mathrm{mixed}}\big(\mathrm{Spec}\,\mathbb{F}_q\big)\big)$.

Finally, we still have $1 \otimes B_w \mapsto (-1)^{\ell(w)}[L^m_w]$.

Proof. Again this morphism is surjective since the standard objects generate the K-group under R, and injective since we may again take the stalks at each point

$$\varphi_w : K^0(D_I(\mathrm{Fl})) \to R, \varphi_w([\mathcal{F}]) = [j^*_w(\mathcal{F})] \in K^0\big(D_{\mathrm{mixed}}\big(\mathrm{Spec}\,\mathbb{F}_q\big)\big)$$

where we may regard $[j^*_w(\mathcal{F})]$ as a mixed sheaf on a point since it is a constant sheaf, so it is an isomorphism of R-module.

Now, for this to be an algebra morphism, we need

$$[j^m_{u*} *_I j^m_{v*}] = [j^m_{uv*}]$$

for $\ell(u) + \ell(v) = \ell(uv)$. This holds by a mixed version of Lemma 2.1.5. Finally, we still need

$$\left([j^m_{u*}] + \left[\overline{\mathbb{Q}}_\ell\left(-\frac{1}{2}\right)\right]\right)\left([j^m_{u*}] - \left[\overline{\mathbb{Q}}_\ell\left(\frac{1}{2}\right)\right]\right) = 0$$

This follows from the exact triangle over $\mathbb{P}^1 = \overline{\mathrm{Fl}_s}$ ($i : \infty \hookrightarrow \mathbb{P}^1$, $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$)

$$i_*i^!\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) \to \overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) \to j_*j^*\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right)$$

Note $j_*j^*\overline{\mathbb{Q}}_\ell[1](1/2) = j^m_{s*}$. Then, note

$$\begin{aligned} i^!\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) &= \mathbb{D}i^*\mathbb{D}\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) \\ &= \mathbb{D}i^*\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) \\ &= \mathbb{D}\overline{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right) \\ &= \overline{\mathbb{Q}}_\ell[-1]\left(-\frac{1}{2}\right) \end{aligned}$$

Thus we arrive at a triangle

$$L^m_s \to j^m_{s*} \to L^m_e\left(-\frac{1}{2}\right)$$

which gives the relation

$$[j^m_{s*}] = [L^m_s] + v$$

so

$$(H_s + v)(H_s - v^{-1}) = [L^m_s]([L^m_s] + v + v^{-1})$$

but by an explicit computation of the convolution $L^m_s *_I L^m_s$ on SL_2, we have the right hand side hold. □

Proposition 2.5.7. *Again, we have*

$$R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} = K^0(D_I(\mathrm{Gr}))$$

with the projection $\mathcal{H} \twoheadrightarrow \mathcal{M}^{\mathrm{sph}}$ corresponding to π_*.

Proof. Likewise as above. □

Proposition 2.5.8. *For any $\lambda \in X_*(T)^+$, the class $[Z^m(V_\lambda)] \in R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}$ is given by*

$$\sum_{\mu \in X_*(T)} \dim V_\lambda^{T^\vee = \mu} \cdot \theta_\mu^m$$

where θ_λ^m is the element in the K-group represented by $[J_\lambda^m]$, and $J_\lambda^m = j_{\mu *}^m *_I j_{\nu !}^m$ is the mixed version of the Wakimoto sheaf.

Proof. Let

$$Q = \sum_{w \in W_f} v^{2\ell(w)}$$

This time, counting in the weights, by the projection formula we have

$$\left[\pi_* \pi^* j_{\lambda !*}^{m,\mathrm{Gr}}\right] = Q \left[j_{\lambda !*}^{m,\mathrm{Gr}}\right]$$

Then, since we have $\pi^* j_{\lambda !*}^{m,\mathrm{Gr}}$ is simple since π is smooth, we have this sheaf of the form $j_{w!*}^m$ for some $w \in W^{\mathrm{ext}}$. Then, we can obviously identify w as the maximal length element with Fl_w mapping in Gr_λ, which we will denote by n_λ. Thus, $\pi^* j_{\lambda !*}^{m,\mathrm{Gr}} = j_{n_\lambda !*}^m$.

Then, note $\pi : \mathrm{Fl}_{n_\lambda} \twoheadrightarrow \mathrm{Gr}_\lambda$ is an affine fibration of dimension $\ell(w_0)$. Thus, we may compute

$$\left[\pi_* \pi^* j_{\lambda !*}^{m,\mathrm{Gr}}\right] = (-v)^{\ell(w_0)} \left[\pi_* j_{n_\lambda !*}^m\right]$$

Thus, we have

$$\left[j_{\lambda !*}^{m,\mathrm{Gr}}\right] = \frac{1}{Q} (-1)^{\ell(n_\lambda) - \ell(w_0)} v^{\ell(w_0)} B_{n_\lambda} \in R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}}$$

But by Kazhdan-Lusztig theory we have

$$B_{n_\lambda} = (-1)^{\ell(n_\lambda) - \ell(w_0)} B_{w_0} \left(\sum_{\mu \in X_*(T)} \dim V_\lambda^{T^\vee = \mu} \cdot \theta_\mu^m \right)$$

But note in $\mathcal{M}^{\mathrm{sph}}$, we have $Q = v^{\ell(w_0)} B_{w_0}$ by a standard computation of L_{w_0}. Hence the result follows from that $Z(\mathcal{H}) \to \mathcal{H} \to \mathcal{M}^{\mathrm{sph}}$ is injective, which is well-known. □

Chapter 3 Constructing and proving the equivalence

Now, we move on to construct the functor and proving the equivalence.

3.1. The de-equivariantization principle

In this subsection, following Achar–Riche[7], we will state the de-equivariantization principle (originally developed by Bezrukavnikov in his work). This will allow us to build up the desired equivalence step by step.

Remark 3.1.1. In short, the de-equivariantization principle answers the following question. Given a reductive group H, a commutative ring A equipped with an H-action, we may define the category $\mathbf{Mod}^H_{A,\mathrm{fr}}$ with objects given by the finite free H-equivariant A -modules (explicitly, modules of the form $V \otimes A^n$ where $V \in \mathbf{Rep}\, H$), and morphisms the H-equivariant morphisms. Now, suppose we have another monoidal Abelian category $\mathcal{A}$ and a monoidal functor $\mathbf{Rep}\, H \to \mathcal{A}$, when can we lift this functor to a monoidal $\mathbf{Mod}^H_{A,\mathrm{fr}} \to \mathcal{A}$? We will first display a toy example that will turn out to be much more useful than it seem.

Construction 3.1.2. We first take $\mathcal{A} = \mathbf{Mod}^K_B$, where $K \subset H$ is a subgroup acting on a ring B. Then, we have a symmetric monoidal functor $\mathbf{Rep}\, H \to \mathbf{Mod}^K_B$ given by $V \mapsto V|_K \otimes B$. We claim that an upgrade of this functor to a symmetric monoidal $\mathbf{Mod}^H_A \to \mathbf{Mod}^K_B$ for some H-equivariant ring A is equivalent to a K-equivariant morphism $A \to B$.

For one, such morphism clearly gives rise to the tensor functor $M \mapsto M \otimes_A B$. For the other, suppose such a lift exist, recall that by the Peter-Weyl theorem we have

$$\mathcal{O}(H) = \bigoplus_{\lambda \in X^*(T)^+} V_\lambda \otimes V_\lambda^* \in \mathrm{Ind}(\mathbf{Rep}\, H)$$

By definition, H-equivariant module is an $\mathcal{O}(H)$-comodule, where the coalgebra structure of $\mathcal{O}(H)$ comes from the group structure of H. Thus, this object satisfies the universal property that

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_A^H)}(M, \mathcal{O}(H) \otimes N) \simeq \mathrm{Hom}_{\mathbf{Mod}_A}(M, N)$$

and in particular we have

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_A^H)}(A, \mathcal{O}(H) \otimes A) = A$$

where this isomorphism respects the algebra structure of A.

Thus, a symmetric monoidal functor $\mathbf{Mod}_A^H \to \mathbf{Mod}_B^K$ asserts firstly that $A \mapsto B$, so upon ind-completing the functor, it induces a morphism

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_A^H)}(A, \mathcal{O}(H) \otimes A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_B^K)}(B, \mathcal{O}(H) \otimes A)$$

On the other hand, the right hand side Hom can be identified with $(\mathcal{O}(H) \otimes A)^K$, so we obtain an equivariant map $A \to B$ by

$$\begin{aligned} A &= \mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_A^H)}(A, \mathcal{O}(H) \otimes A) \\ &\to \mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}_B^K)}(B, \mathcal{O}(H) \otimes B) \\ &= (\mathcal{O}(H) \otimes B)^K \\ &\to (\mathcal{O}(K) \otimes B)^K = B \end{aligned}$$

the two constructions are evidently inverse to each other.

Construction 3.1.3. We then treat the general case where we wish to find the monoidal lift $\tilde{F}$ of the symmetric monoidal functor F

$$\begin{tikzcd} \mathbf{Rep}\, H \arrow[r, "F"] \arrow[d] & \mathcal{A} \\ \mathbf{Mod}_{A,\mathrm{fr}}^H \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

where $\mathcal{A}$ is assumed merely to be monoidal. The method is to reduce to the previous case. Since $\mathbf{Rep}(H)$ is symmetric monoidal, the commutativity constraint $V_1 \otimes V_2 \simeq V_2 \otimes V_1$ is sent by F to a commutativity constraint $F(V_1) \otimes F(V_2) \simeq F(V_2) \otimes F(V_1)$. Then, we construct a faithful functor of monoidal Abelian categories of $\mathcal{C} \subset \mathcal{A}$. We define $\mathcal{C}$ to be the category with objects the same as $\mathbf{Rep}\, H$, and for objects V, V', we define $\mathrm{Hom}_{\mathcal{C}}(V, V') \subset \mathrm{Hom}_{\mathcal{A}}(F(V), F(V'))$ to be the subgroup of morphisms f such that for any object V'' the following diagram commutes

$$\begin{tikzcd}
F(V'') \otimes F(V) \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\simeq"'] & F(V'') \otimes F(V') \arrow[d, "\simeq"] \\
F(V) \otimes F(V'') \arrow[r, "f \otimes \mathrm{id}"] & F(V') \otimes F(V'')
\end{tikzcd}$$

We first attempt to identify this category $\mathcal{C}$. Notice that $\mathcal{C}$ inherits the monoidal structure from $\mathcal{A}$, and moreover the above commutative square makes $\mathcal{C}$ a symmetric monoidal category. We claim that $\mathcal{C}$ can be identified as a symmetric monoidal category with $\mathbf{Mod}^H_{B,\mathrm{fr}}$ for some B. Firstly, we have the symmetric monoidal functor $F : \mathbf{Rep}(H) \to \mathcal{A}$ factors through the inclusion $\mathcal{C} \hookrightarrow \mathcal{A}$, in which we denote the factoring by $F_{\mathcal{C}} : \mathbf{Rep}\, H \to \mathcal{C}$. We define

$$B = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(1_{\mathcal{C}}, F_{\mathcal{C}}(\mathcal{O}(H)))$$

Then, we may construct a functor $H : \mathcal{C} \to \mathbf{Mod}^H_{B,\mathrm{fr}}$ given by

$$H(V) \coloneqq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(1_{\mathcal{C}}, F_{\mathcal{C}}(V) \otimes F_{\mathcal{C}}(\mathcal{O}(H)))$$

By moving $F_{\mathcal{C}}(V) \otimes F_{\mathcal{C}}(\mathcal{O}_H(H)) = F_{\mathcal{C}}(V \otimes \mathcal{O}(H))$, we may twist the H-action on V into $\mathcal{O}(H)$ through an isomorphism of H-modules. Thus, we may assume V is a trivial H-module, and we move V to the outside and we may identify $H(V) = V \otimes B$ (as a direct corollary, H is essentially surjective). With the same process, we see that H is fully faithful. By unwinding the definitions, we see H is monoidal.

Hence, we have the data of a lift $\tilde{F}$ coming from an H-equivariant map $A \to B$ for B defined above.

Remark 3.1.4. Actually, the full content of the de-equivariantization principle extends further. Recall that we may define for a ring A the concept of A-linear categories. We may extend this definition to $\mathcal{X}$-linear categories, where $\mathcal{X}$ is a general stack. Notice that a monoidal functor of monoidal categories $\mathbf{Rep}\, H \to \mathcal{A}$ is equivalent to a $\mathbb{B}H$-linear structure on $\mathcal{A}$ (here $\mathbb{B}H$ is the Artin stack of the delooping of H). Then, an extension to $\mathbf{Mod}^H_{A,\mathrm{fr}} \to \mathcal{A}$ is equivalent to a $((\mathrm{Spec}\, A)/H)$-linear structure on $\mathcal{A}$.

So, one may expect that the de-equivariantization principle would provide the equivalent piece of data of upgrading a $\mathbb{B}H$-linear category into a $((\mathrm{Spec}\, A)/H)$-linear category. Explicitly, this piece of data is an H-equivariant A-linear structure on $\mathcal{A}_{\mathrm{deeq}}$, where $\mathcal{A}_{\mathrm{deeq}}$ is the $\mathcal{O}(H)$-algebras in $\mathcal{A}$.

3.2. The construction of the functor Φ

We first give a brief outline of the construction of the equivalence Φ (the functor in Theorem 1.2.1).

Construction 3.2.1. We wish to construct a functor originating from $D\mathrm{Coh}^{G^\vee}(\tilde{N}^\vee)$. Recall that the Springer resolution $\tilde{N}^\vee$ on the dual side is given by

$$\mathcal{B}^\vee \times \mathfrak{g}^\vee \supset \tilde{N}^\vee = \{(\mathfrak{b}, x) \in B^\vee \times \mathfrak{g}^\vee | x \in \mathfrak{b}\}$$

However, $\tilde{N}^\vee$ is the total space of a vector bundle over a projective space, hence highly projective in nature. This blocks our way to analyse the structure of its category of coherent sheaves directly.

The way to get past this is considering the $T^\vee$-torsor $G^\vee/U^\vee \twoheadrightarrow \mathcal{B}^\vee$. It is well-known that $G^\vee/U^\vee$ is quasi-affine. By the Peter-Weyl theorem, the affine closure of $G^\vee/U^\vee$ is given by

$$\overline{G^\vee/U^\vee} = \operatorname{Spec} \mathcal{O}(G^\vee/U^\vee) = \operatorname{Spec}\left(\bigoplus_{\lambda \in X_*(T)^+} V_\lambda\right)$$

Here, we give a commutative ring structure on $\bigoplus V_\lambda$. Firstly, the underlying k-vector space structure of this ring is the direct sum of all finite dimensional $G^\vee$-representations. Then, notice that by the formalism of highest weight representations, we have a non-trivial morphism (up to scalar, so we fix one compatible with each other)

$$m_{\lambda+\rho} : V_\lambda \otimes V_\rho \to V_{\lambda+\rho}$$

which is defined as follows: notice that the highest weight line $b_\lambda \subset V_\lambda$ and the highest weight line $b_\rho \subset V_\rho$ gives a highest weight line $b_\lambda \otimes b_\rho \subset V_\lambda \otimes V_\rho$, and hence a factor $V_{\lambda+\rho} \hookrightarrow V_\lambda \otimes V_\rho$. Since the category of $G^\vee$-representations are semisimple, we get a projection $V_\lambda \otimes V_\rho \to V_{\lambda+\rho}$.

We first let the preimage of $\widetilde{\mathcal{N}}^\vee$ in $G^\vee/U^\vee \times \mathfrak{g}^\vee$ be $\widehat{\mathcal{N}}^\vee$. Then, since $\widehat{\mathcal{N}}^\vee$ is closed in $G^\vee/U^\vee \times \mathfrak{g}^\vee$, consider the ideal sheaf associated to it, say $\mathcal{I} \subset \mathcal{O}_{G^\vee/U^\vee \times \mathfrak{g}^\vee}$. Then,

$\mathcal{J}(G^\vee/U^\vee \times \mathfrak{g}^\vee) \subset \mathcal{O}(G^\vee/U^\vee \times \mathfrak{g}^\vee)$ is an ideal, which gives rise to a closed subscheme of $\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee$, which we call $\widehat{\mathcal{N}}^\vee_{\text{aff}}$. Clearly, we have

$$\widehat{\mathcal{N}}^\vee_{\text{aff}} \times_{\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee} (G^\vee/U^\vee \times \mathfrak{g}^\vee) = \widehat{\mathcal{N}}^\vee$$

and the inclusion $\widehat{\mathcal{N}}^\vee \subset \widehat{\mathcal{N}}^\vee_{\text{aff}}$ exhibits $\widehat{\mathcal{N}}^\vee$ as a quasi-affine scheme. We denote its boundary by $\partial\widehat{\mathcal{N}}^\vee := \widehat{\mathcal{N}}^\vee_{\text{aff}} - \widehat{\mathcal{N}}^\vee$.

On the other hand, the category of perverse sheaves $\mathbf{Perv}_I(\text{Fl})$ is not monoidal, since convolution takes objects out. We must replace it by the monidal category $\mathbf{Perv}_I^{\text{waki}}(\text{Fl})$, which is the subcategory of $\mathbf{Perv}_I(\text{Fl})$ consisting of objects admitting a filtration by Wakimoto sheaves. This category is monoidal since $J_\lambda *_I J_\mu = J_{\lambda+\mu}$, and it is still large enough to contain the image of the central functor Z.

With the above definitions, our way to define the functor Φ can be summarized as follows:

(1) Find a monoidal functor

$$\Phi_0 : \mathbf{Rep}(G^\vee \times T^\vee) \to \mathbf{Perv}_I^{\text{waki}}(\text{Fl})$$

and find the factoring through the faithful subcategory $\mathcal{C} \subset \mathbf{Perv}_I^{\text{waki}}(\text{Fl})$ (constructed in the previous section, proven to be equivalent to a module category), whose induced functor $\mathbf{Rep}(G^\vee \times T^\vee) \to \mathcal{C}$ is symmetric monoidal.

(2) Extend it via the de-equivariantization principle to

$$\Phi_1 : \text{Coh}_{\text{fr}}^{G^\vee \times T^\vee}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathcal{C} \hookrightarrow \mathbf{Perv}_I^{\text{waki}}(\text{Fl})$$

(3) Restrict again via the de-equivariantization principle to

$$\Phi_2 : \text{Coh}_{\text{fr}}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\text{aff}}\right) \to \mathcal{C} \hookrightarrow \mathbf{Perv}_I^{\text{waki}}(\text{Fl})$$

(4) Post-compose the inclusion $\mathbf{Perv}_I^{\text{waki}}(\text{Fl}) \hookrightarrow \mathbf{Perv}_I(\text{Fl})$, and take homotopy categories. We have by definition

$$K\Big(\text{Coh}_{\text{fr}}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\text{aff}}\right)\Big) = \text{Perf}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\text{aff}}\right)$$

and for the regular stack $\widehat{\mathcal{N}}^\vee/(G^\vee \times T^\vee)$, we have

$$\text{Perf}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\text{aff}}\right) = D\text{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\text{aff}}\right)$$

Hence we arrive at a map

$$K(\Phi_2) : D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to K(\mathbf{Perv}_I(\mathrm{Fl})) \twoheadrightarrow D(\mathbf{Perv}_I(\mathrm{Fl}))$$

We then show the complexes of sheaves supported on $\partial\widehat{\mathcal{N}}^\vee$ are sent by Φ_3 to acyclic complexes, so we obtain a functor

$$\begin{aligned}\Phi_3 : D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right)/D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right)_{\partial\widehat{\mathcal{N}}^\vee} &= D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee\right) \\ &\to D(\mathbf{Perv}_I(\mathrm{Fl}))\end{aligned}$$

But we have by definition

$$D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee\right) = D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)$$

Then by projecting to ${}^f\mathcal{P}_I$, the construction of Φ is done, although we will more often use the functor with image $D(\mathbf{Perv}_I(\mathrm{Fl}))$ rather than the one with image $D\left({}^f\mathcal{P}_I\right)$.

We then exhibit the four steps stated above more clearly.

In these steps, we make more use of the Wakimoto sheaves, especially the Wakimoto filtrations, proven to exist for central convolution exact sheaves, for example the image of Z, as in Proposition 2.3.1.

Definition 3.2.2. Firstly, we let $\mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$ denote the subcategory of $\mathbf{Perv}_I(\mathrm{Fl})$ consisting of objects admitting a filtration by Wakimoto sheaves. By Lemma 2.3.2, the filtration on each $\mathcal{F} \in \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$ is unique, so we may define an associated graded functor

$$\mathrm{gr}_{\mathrm{waki}} : \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl}) \to \mathbf{Rep}\, T^\vee$$

since we recall that the Wakimoto sheaves are indexed by coweights. Moreover, since $J_\lambda *_I J_\mu = J_{\lambda+\mu}$, so $\mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl}) \subset D_I(\mathrm{Fl})$ inherits the monoidal structure.

Construction 3.2.3. We construct the functor

$$\Phi_0 : \mathbf{Rep}(G^\vee \times T^\vee) \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$$

Recall in Theorem 2.2.1 and Proposition 2.3.1 we have constructed a functor Z : $\mathbf{Perv}_{L^+G}(\mathrm{Gr}) \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$. Precomposing this with the Satake functor gives

$$\mathbf{Rep}\, G^\vee \xrightarrow{\mathrm{Sat}_G} \mathbf{Perv}_{L^+G}(\mathrm{Gr}) \xrightarrow{Z} \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$$

The Satake functor is symmetric monoidal and the central functor is central and monoidal, so this is monoidal. We write $Z(\mathrm{Sat}_G(V_\lambda)) := Z_\lambda$ for short. On the other hand, by Lemma 2.1.13, we have the Wakimoto sheaves functor

$$\mathbf{Rep}\,T^\vee \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl}),\, \left(\overline{\mathbb{Q}}_\ell\right)_\lambda \mapsto J_\lambda$$

is monoidal. Thus, the two functors combines and give a functor

$$\mathbf{Rep}(G^\vee \times T^\vee) \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl}),\, V_\lambda \otimes \left(\overline{\mathbb{Q}}_\ell\right)_\eta \mapsto Z_\lambda *_I J_\eta$$

This functor is monoidal since Z is in addition central.

Construction 3.2.4. We now construct the functor

$$\Phi_1 : \mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$$

Here, we first specify the $(G^\vee \times T^\vee)$-action on $\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee$. The torus $T^\vee$ acts on $\overline{G^\vee/U^\vee}$ by multiplying on the right, and on $\mathfrak{g}^\vee$ trivially. The group $G^\vee$ acts on $\overline{G^\vee/U^\vee}$ on the left (or, equivalently, the resulting action on the $G^\vee/B^\vee$-part of the Springer resolution is given by $g \cdot \mathfrak{b}^\vee = \mathrm{Ad}_g\, \mathfrak{b}^\vee$), and the adjoint action on $\mathfrak{g}^\vee$.

Again, we construct the two functors separately. Firstly, we construct a functor

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee}(\mathfrak{g}^\vee) \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$$

Recall by Tannakian formalism, the functor of points of $\mathfrak{g}^\vee$ can be identified with

$$\mathfrak{g}^\vee \otimes A = \mathrm{End}_\otimes\left(\mathbf{Rep}\,G^\vee \xrightarrow{-\otimes A} \mathbf{Mod}_A\right)$$

where $\mathrm{End}_\otimes$ denotes the set of tensor endomorphisms or tensor derivations of the fiber functor. Hence, if A is equipped with a $G^\vee$-action, a $G^\vee$-equivariant map of algebras $\mathcal{O}(\mathfrak{g}^\vee) \to A$ is equivalent to a $G^\vee$-equivariant map of k-vector spaces $(\mathfrak{g}^\vee)^* \to A$, hence an element in $(\mathfrak{g}^\vee \otimes A)^{G^\vee}$. This is simply a $G^\vee$-equivariant tensor endomorphism of the functor $\mathbf{Rep}\,G^\vee \to \mathbf{Mod}_A$.

In our setting, we have a functor

$$\mathbf{Rep}\,G^\vee \to \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl})$$

equipped with a G-equivariant tensor derivation given by M as in Theorem 2.2.1. Hence, by the formalism introduced in Construction 3.1.3, we may first restrict to the subcategory of ${}^f\mathcal{P}_I$ of the form $\mathbf{Mod}^{G^\vee}_{B,\mathrm{fr}}$, hence by the above argument lifting to a functor $\mathrm{Coh}^{G^\vee}_{\mathrm{fr}}(\mathfrak{g}^\vee) \to {}^f\mathcal{P}_I$.

We then treat the more difficult case. Now, let A be a ring equipped with an action of $G^\vee \times T^\vee$, that is, A has a grading by $X_*(T)$ and a $G^\vee$-action preserving the grading. Then, a $(G^\vee \times T^\vee)$-equivariant morphism $\mathcal{O}\left(\overline{G^\vee/U^\vee}\right) \to A$ is a $G^\vee$-equivariant map

$$\bigoplus_{\lambda \in X_*(T)^+} V_\lambda \to A$$

compatible with the grading of A. The data of $V_\lambda \to A$ compatible with grading is equivalent to a $T^\vee$-equivariant map $b_\lambda : V_\lambda \otimes A \to A(\lambda)$, where $A(\lambda)$ is A with the action of $T^\vee$ twisted by λ, explicitly

$$A(\lambda)_\mu = A_{\lambda+\mu}$$

Moreover, for the above map to be a ring homomorphism, these maps must satisfy the Plücker relations. Namely, we should have

$$b_\lambda \otimes b_\mu = b_{\lambda+\mu}(m \otimes \mathrm{id}_A) : (V_\lambda \otimes A) \otimes_A \left(V_\mu \otimes A\right) \to A(\lambda+\mu)$$

Thus, again using Construction 3.1.3, a functor

$$\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}\left(\overline{G^\vee/U^\vee}\right) \to \mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl})$$

amounts to mapping the arrows

$$B_\lambda : V_\lambda \otimes \mathcal{O} \to \mathcal{O}(\lambda)$$

to arrows satisfying the Plücker relations. Here, $V_\lambda \otimes \mathcal{O}$ corresponds to the trivial vector bundle with fiber V_λ, equipped with the diagonal G-action. $\mathcal{O}(\lambda)$ is the vector bundle corresponding to

$$\bigoplus_{\mu \in X_*(T)^+} V_{\lambda+\mu}$$

Since we know $V_\lambda \otimes \mathcal{O} \mapsto Z_\lambda$, and $\mathcal{O}(\lambda) \mapsto J_\lambda = j_{t_\lambda *}$, it suffices to find maps $b_\lambda : Z_\lambda \to j_{t_\lambda *}$ satisfying the Plücker relations. This will be done in the following proposition Proposition 3.2.6.

In total, we obtain the functor

$$\Phi_1 : \mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl})$$

Lemma 3.2.5. *For all $\lambda \in X_*(T)^+$, we have the cell Fl_{t_λ} open in $\mathrm{Supp}\, Z_\lambda$, and we have an isomorphism $j_{t_\lambda}^*(Z_\lambda) = \overline{\mathbb{Q}}_\ell[\ell(t_\lambda)]$.*

Proof. Recall $Z_\lambda = Z\big(L_\lambda^{\mathrm{Gr}}\big)$, and by the nearby cycles construction, we see

$$\mathrm{Supp}\, Z_\lambda \supset \pi^{-1}\Big(\mathrm{Gr}_{t_\lambda}\Big)$$

Then, since Gr_λ is also of dimension $\ell(t_\lambda)$, so $\mathrm{Fl}_{t_\lambda} \subset \pi^{-1}(\mathrm{Gr}_\lambda)$. Moreover, since the nearby cycles functor preserves the dimension of the support, we have

$$\dim \mathrm{Supp}\, Z_\lambda = \dim \mathrm{Gr}_\lambda = \ell(t_\lambda) = \dim \mathrm{Fl}_{t_\lambda}$$

and since Fl_{t_λ} has dimension equal to the support, it is open. Then, since

$$\pi : \mathrm{Fl}_{t_\lambda} \hookrightarrow \mathrm{Gr}_\lambda$$

is injective and open dense, by Theorem 2.2.1(4) we can identify the stalk $j_{t_\lambda}^*(Z_\lambda)$ as the stalk of L_λ^{Gr} on Gr_λ. This is clearly $\overline{\mathbb{Q}}_\ell[\ell(t_\lambda)]$. □

Proposition 3.2.6. *By the previous lemma, we obtain a map up to scalar*

$$b_\lambda : Z_\lambda \to j_{t_\lambda *}$$

for all $\lambda \in X_*(T)^+$. These maps satisfy the Plücker relation. Namely, we have the commutative diagram (up to scalar)

$$\begin{tikzcd}
Z_{\lambda+\mu} \arrow[r, "Z(m_{\lambda,\mu})"] \arrow[d, "b_{\lambda+\mu}"'] & Z_\lambda *_I Z_\mu \arrow[d, "b_\lambda *_I b_\mu"] \\
j_{t_{\lambda+\mu}*} \arrow[r, "\simeq"] & j_{t_\lambda *} *_I j_{t_\mu *}
\end{tikzcd}$$

Proof. Since $\pi : \mathrm{Fl}_\lambda \to \mathrm{Gr}_\lambda$ is an open immersion, π_* induces a fully faithful embedding on the category of sheaves. Hence, we have

$$\mathrm{Hom}_{\mathbf{Perv}_I(\mathrm{Fl})}\Big(Z_\lambda, j_{t_\lambda *}\Big) = \mathrm{Hom}_{\mathbf{Perv}_{L^+G}(\mathrm{Gr})}\Big(L_{t_\lambda}^{\mathrm{Gr}}, j_{t_\lambda *}^{\mathrm{Gr}}\Big) = \overline{\mathbb{Q}}_\ell$$

so such $b_{\lambda+\mu}$ is canonical up to scalar. Thus, the above diagram commute up to scalar, and we can fix one such that the diagram really commute. □

We then proceed to the next step.

Construction 3.2.7. We then upgrade Φ_1 to a functor

$$\Phi_2 : \mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl})$$

This step is relatively simple. The previous functor

$$\Phi_1 : \mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl})$$

comes from an equivariant map of rings

$$\mathcal{O}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to B$$

where B is obtained as in the general formalism introduced in Construction 3.1.3. So, it suffices to prove the functions defining the closed subscheme $\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}$ is killed by the above map. This corresponds to the condition $b_\lambda \circ M_{Z_\lambda} = 0$, since the closed immersion $\widehat{\mathcal{N}}^\vee_{\mathrm{aff}} \subset \overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee$ can be characterized similarly. Note that b_λ is adjoint to the identity map of $\overline{\mathbb{Q}}_\ell[\ell(t_\lambda)]$ on Fl_λ. Then, $b_\lambda \circ M_{Z_\lambda} : Z_\lambda \to Z_\lambda \to j_{t_\lambda *}$ is adjoint to $j^*_{t_\lambda}\left(M_{Z_\lambda}\right)$, so it suffices to prove $j^*_{t_\lambda}\left(M_{Z_\lambda}\right) = 0$. This is because this map is nilpotent, but

$$\mathrm{End}\left(j^*_{t_\lambda}(Z_\lambda)\right) = \mathrm{End}\left(\overline{\mathbb{Q}}_\ell[\ell(t_\lambda)]\right) = \overline{\mathbb{Q}}_\ell$$

is 1-dimensional, so this map must be 0.

Thus, we have finished the construction of Φ_2.

We now begin to do the final reduction.

Lemma 3.2.8. *The functor* $\mathrm{gr}_{\mathrm{waki}}$ *is monoidal.*

Proof. The monoidality follows from the general formalism of taking associated graded objects of filtered objects in Abelian categories. ☐

Proposition 3.2.9. *The composition*

$$\mathbf{Rep}\, G^\vee \xrightarrow{Z} \mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl}) \xrightarrow{\mathrm{gr}_{\mathrm{waki}}} \mathbf{Rep}\, T^\vee$$

is equivalent to the restriction map along the embedding $T^\vee \hookrightarrow G^\vee$

Proof. This will follow from the Tannakian formalism. We first show that $\mathrm{gr}_{\mathrm{waki}} \circ Z$ is a symmetric monoidal functor. Recall Z is central, and $\mathrm{gr}_{\mathrm{waki}}$ admits a right inverse given by the direct sum of Wakimoto sheaves. Then, let $\sigma_{X,Y} : Z(X) *_I Y \to Y *_I Z(X)$ denote the centrality isomorphism. Since $\mathrm{gr}_{\mathrm{waki}}$ admits a right inverse, $\mathrm{gr}_{\mathrm{waki}}\, \sigma_{X,Y}$ is a centrality isomorphism for $\mathrm{gr}_{\mathrm{waki}} \circ Z$ because the naturality diagrams can be checked in $\mathbf{Perv}^{\mathrm{waki}}_I(\mathrm{Fl})$.

Now, we have $\mathrm{gr}_{\mathrm{waki}} \circ Z$ is a central monoidal functor, and Z is in addition symmetric, so it can be checked that the centrality isomorphism of $\mathrm{gr}_{\mathrm{waki}} \circ Z$ agrees with the symmetric monoidal structure on $\mathbf{Rep}\, T^\vee$ by the Plücker relations. Thus, we have $\mathrm{gr}_{\mathrm{waki}} \circ Z$ is symmetric monoidal, and hence the functor is the restriction functor of some morphism of algebraic groups $T^\vee \to G^\vee$. Again, by the highest weight line b_λ for $\lambda \in X_*(T)^+$, we see that $\mathrm{gr}_{\mathrm{waki}} \circ Z(V_\lambda)$ contains exactly one copy of k_λ, hence $T^\vee \to G^\vee$ is injective. $\square$

This allows us to prove the following proposition.

Proposition 3.2.10. *By construction, the image of Φ_2 lands in the subcategory of Wakimoto sheaves. The composition*

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \xrightarrow{\Phi_2} \mathbf{Perv}_I^{\mathrm{waki}}(\mathrm{Fl}) \xrightarrow{\mathrm{gr}_{\mathrm{waki}}} \mathbf{Rep}\, T^\vee$$

agrees with the pullback of coherent sheaves along the point $\iota : (1, 0) \hookrightarrow \widehat{\mathcal{N}}^\vee_{\mathrm{aff}}$. Here $(1, 0)$ denotes the $T^\vee$-fixed point in $\widehat{\mathcal{N}}^\vee_{\mathrm{aff}} = G^\vee \times^{U^\vee} \mathfrak{n}^\vee$ given by the base point of the vector bundle over $1 \in G^\vee/U^\vee$.

Proof. Firstly recall that precomposing this functor with

$$\mathbf{Rep}(G^\vee \times T^\vee) \to \mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right)$$

gives the restriction along the diagonal map $T^\vee \hookrightarrow G^\vee \times T^\vee$. Hence, we may identify the functor in question with the functor obtained via Construction 3.1.2 through a $(G^\vee \times T^\vee)$ -equivariant morphism of rings

$$\mathcal{O}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \overline{\mathbb{Q}}_\ell$$

so it suffices to identify this morphism with the point $(1, 0)$. Then, to determine the coordinates of this point in $\mathfrak{g}$ and $\overline{G^\vee/U^\vee}$, respectively, we must inspect where $\mathrm{gr}_{\mathrm{waki}}$ sends the morphisms $M_{Z_\lambda} : Z_\lambda \to Z_\lambda$ and $b_\lambda : Z_\lambda \to j_{t_\lambda *}$ to.

We first claim that the first nilpotent endomorphism is sent to 0. This is because by Proposition 2.1.10, we can identify the cohomology of a Wakimoto-filtered sheaf $\mathcal{F}$ with the cohomology of $\mathrm{gr}_{\mathrm{waki}}(\mathcal{F})$ by the parity vanishing of the cohomology of Wakimoto sheaves. So, to prove that M_{Z_λ} is sent to 0, it suffices to prove it is 0 on cohomology, since it can only interpolate between the same graded piece, as can be seen from the action on cohomology. Then, this map is 0 since $H^*(\mathrm{Fl}, Z_\lambda)$ can be computed by its proper pushforward to Gr, and the logarithm of the monodromy map on $H^*(\mathrm{Gr}, \pi_* Z_\lambda)$ can be easily shown to be zero, since this monodromy can be obtained by a nearby cycles functor for the Beilinson-Drinfeld Grassmannian over a curve, but the nearby cycles functor itself is the identity map, which gives the identity monodromy, whose logarithm is 0.

For the second highest weight line, its associated graded is the projection onto the highest weight factor.

Hence, the point corresponding to the above equivariant morphism is exactly $(1, 0)$.□

Construction 3.2.11. Now we may finally construct the final functor

$$\Phi_3 : D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee\right) \to D(\mathbf{Perv}_I(\mathrm{Fl}))$$

As in Construction 3.2.1, we first take homotopy categories, and we obtain a functor

$$K(\Phi_2) : K\left(\mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)\right) = D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right) \to K(\mathbf{Perv}_I(\mathrm{Fl})) \twoheadrightarrow D(\mathbf{Perv}_I(\mathrm{Fl}))$$

Since we have (this is a version of Beuville-Laszlo's theorem)

$$D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right) / D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)_{\partial \widehat{\mathcal{N}}^\vee} = D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}^\vee\right)$$

it suffices to prove the above morphism $K(\Phi_2)$ vanishes on $D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)_{\partial \widehat{\mathcal{N}}^\vee}$, so by the universal property of Serre quotients we obtain the final functor.

Since taking associated graded pieces is conservative, it suffices to prove that for any $\mathcal{F} \in D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)_{\partial \widehat{\mathcal{N}}^\vee}$, we have $\mathrm{gr}\, K(\Phi_2)(\mathcal{F}) \in D\,\mathbf{Rep}(T^\vee)$ is zero. This is because taking $\mathrm{gr}\, K(\Phi_2)$ amounts to taking the derived pullback

$$\iota^* : D\mathrm{Coh}^{G^\vee \times T^\vee}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right) \to D\,\mathbf{Rep}(T^\vee)$$

which certainly vanishes on sheaves supported on $\partial \widehat{\mathcal{N}}^\vee$.

Thus we have finished the construction of the functor Φ_3, which we will now denote by Φ.

3.3. Tilting sheaves

Now, recall from the previous section that we must prove Theorem 2.4.1 and Theorem 2.4.2 instead.

Definition 3.3.1. We define the functor

$$\Phi_{\mathcal{IW}} = \mathrm{Av}_\Psi \circ \Phi$$

to be the composition

$$D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \to D(\mathbf{Perv}_I(\mathrm{Fl})) \to D_{\mathcal{IW}}(\mathrm{Fl})$$

we remark that in particular this functor factors through the derived category $D({}^f\mathcal{P}_I)$, since the averaging functor $\mathrm{Av}_\Psi : \mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$ factors through the fully faithful functor ${}^f\mathcal{P}_I \hookrightarrow \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$. This factoring will be used to the calculation of certain Hom spaces.

We also define the functor

$$Z_{\mathcal{IW}} = \mathrm{Av}_\Psi \circ Z$$

to be the composition

$$\mathbf{Rep}\, G^\vee \to \mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$$

Definition 3.3.2. We fix an interval finite poset Λ, where Λ is interval finite if for any $\nu \leq \mu \in \Lambda$, we have $\{\lambda \in \Lambda \mid \nu \leq \lambda \leq \mu\}$ is a finite set. In particular, the quasi-coxeter group W^{ext} with the Bruhat order is interval finite. We equip Λ with the order topology (closed sets are the sets closed under taking smaller objects).

We define a highest weight category to be an k-linear Abelian category $\mathcal{A}$ with the following properties:

(1) $\mathcal{A}$ is a finite length category.
(2) The set of simple objects $\mathrm{IC}(\lambda)$ is indexed by Λ. Moreover, we require there are some morphisms $\Delta(\lambda) \to \mathrm{IC}(\lambda)$ and $\mathrm{IC}(\lambda) \to \nabla(\lambda)$. They are called standard and costandard objects respectively.
(3) $\mathrm{End}(\mathrm{IC}(\lambda)) = k$.
(4) We let $I \subset \Lambda$ be closed. We define $\mathcal{A}_I$ to be the subcategory spanned by the simple objects $\mathrm{IC}(i)$ for $i \in I$. We let $i_0 \in I$ be a maximal element. Then we have the structure morphism $\Delta(i_0) \to \mathrm{IC}(i_0)$ is a projective cover and $\mathrm{IC}(i_0) \to \nabla(i_0)$ is an injective hull.
(5) $\ker(\Delta(\lambda) \to \mathrm{IC}(\lambda))$ and $\mathrm{coker}(\mathrm{IC}(\lambda) \to \Delta(\lambda))$ lies in $\mathcal{A}_{<\lambda}$.
(6) $\mathrm{Ext}^2(\Delta(\lambda), \nabla(\lambda)) = 0$.

We define a tilting object a in a highest weight category to be an object equipped with both a filtration by standard objects and a filtration by costandard objects. We define the number $(a : \Delta(\lambda))$ to be the multiplicity of $\Delta(\lambda)$ in the standard filtration, and $(a : \nabla(\lambda))$ likewise. These numbers are actually well-defined.

Remark 3.3.3. By the formalisms introduced in the paper[12], we can show that $\mathbf{Perv}_{\mathcal{JW}}(\mathrm{Fl})$ is a highest weight category, with standard, simple, and costandard objects given by the obvious sheaves. This is a relatively simple case, and many properties of highest weight categories follow from the functors $i_{w!}$, i_{w*}, $i_w^!$, and i_w^* associated to the stratification. In particular, the computation $\mathrm{RHom}\big(\Delta_\lambda, \nabla_\mu\big) = \overline{\mathbb{Q}}_\ell$ for $\lambda = \mu$ and 0 otherwise follows from these adjunctions. This proves the well-definedness of $(\mathcal{F} : \Delta_\lambda)$ and $(\mathcal{F} : \nabla_\lambda)$ in this case, since they are computed by $\mathrm{Hom}(\mathcal{F}, \nabla_\lambda)$ and $\mathrm{Hom}(\Delta_\lambda, \mathcal{F})$.

We then present the most important proposition for the proof of the equivalence Theorem 2.4.1.

Proposition 3.3.4. *For any $V \in \mathbf{Rep}\, G^\vee$, the perverse sheaf $Z_{\mathcal{JW}}(V)$ is tilting, and for any $\lambda \in X^\vee$, we have*

$$(Z_{\mathcal{JW}}(V) : \Delta_\lambda) = (Z_{\mathcal{JW}}(V) : \nabla_\lambda) = \dim\Big(V^{T^\vee = \lambda}\Big)$$

We will first prove the above proposition assuming $Z_{\mathcal{JW}}(V)$ is tilting, then prove the assertion.

Lemma 3.3.5. *The multiplicities in Proposition 3.3.4 can be computed if we assume $Z_{\mathcal{JW}}(V)$ is tilting.*

Proof. We prove the following equalities for $V \in \mathbf{Rep}\, G^\vee$:

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\big(\Delta_\mu, Z_{\mathcal{JW}}(V)[i]\big) = \dim\Big(V^{T^\vee = \mu}\Big)$$

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\big(Z_{\mathcal{JW}}(V)[i], \nabla_\mu\big) = \dim\Big(V^{T^\vee = \mu}\Big)$$

Notice that the Euler characteristic as indicated above can be computed in terms of the K -group, so by Proposition 2.5.5, we have

$$[Z(V)] = \sum_{\lambda \in X_*(T)} \dim\Big(V^{T^\vee = \lambda}\Big)[J_\lambda]$$

Then, since $[j_{s*}] = [j_{s!}]$ in the K-group, writing $\lambda = \mu - \rho$ for $\mu, \rho \in X_*(T)^+$, we have

$$[J_\lambda] = \left[j_{t_\mu *} *_I j_{t_{(-\rho)!}}\right] = \left[j_{t_\mu *}\right]\left[j_{t_{-\rho} *}\right] = \left[j_{t_{\mu-\rho} *}\right] = \left[j_{t_\lambda *}\right]$$

by Proposition 2.5.2.

To sum up, we have

$$[Z(V)] = \sum_{\lambda \in X_*(T)} \dim\left(V^{T^\vee = \lambda}\right)\left[j_{t_\lambda *}\right]$$

and hence since Av_Ψ is exact, we have

$$[Z_{\mathcal{IW}}(V)] = \sum_{\lambda \in X_*(T)} \dim\left(V^{T^\vee = \lambda}\right)[\nabla_\lambda]$$

by Proposition 2.4.9. The claim follows by highest weight category formalism, since by the Hom orthogonality and Ext vanishing, we have

$$\sum_{i \geq 0} (-1)^i \dim\left(\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_\mu, \nabla_\nu)\right) = \delta_{\mu,\nu}$$

Thus we have the first equality. The second is similar. Note the computation of the Euler characteristic do not need the assumption that $Z_{\mathcal{IW}}(V)$ is tilting.

Then, we use these equalities to compute the multiplicity $(Z_{\mathcal{IW}}(V) : \Delta_\lambda)$, and the other is similar. Now, by the tilting assumption, we have

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_\lambda, Z_{\mathcal{IW}}(V)[i]) = 0$$

by Ext vanishing and taking the costandard filtration for $Z_{\mathcal{IW}}(V)$. On the other hand, by highest weight formalism, since $\mathrm{Hom}(\Delta_\lambda, \nabla_\lambda) = \overline{\mathbb{Q}}_\ell$, we have

$$\dim \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_\lambda, Z_{\mathcal{IW}}(V)) = (Z_{\mathcal{IW}}(V) : \nabla_\lambda)$$

so as a corollary we have

$$(Z_{\mathcal{IW}}(V) : \nabla_\lambda) = \dim\left(V^{T^\vee = \lambda}\right)$$

the other equality follows dually. □

Now, it suffices to prove that $Z_{\mathcal{IW}}(V)$ is tilting. This is difficult, and we will achieve this through the following lemmas.

Lemma 3.3.6. *If $V, V' \in \mathbf{Rep}\, G^\vee$ is such that $Z_{\mathcal{IW}}(V)$ and $Z_{\mathcal{IW}}(V')$ are tilting, then $Z_{\mathcal{IW}}(V \otimes V')$ is tilting.*

Lemma 3.3.7. *Assume that G is semisimple. If V is a highest weight $G^\vee$-module whose highest weight is either minuscule or quasi-minuscule, then $Z_{\mathcal{JW}}(V)$ is tilting.*

Proof. We first apply the lemmas above to prove Proposition 3.3.4. Notice that obviously we may assume V is simple. Then, for $G = G' \times T'$, where T' is a central torus and G' semisimple, Gr_G is a union of $\mathrm{Gr}_{G'}$ indexed by $X_*(T')$. The same follows for Fl_G and $\mathrm{Fl}_{G'}$. Thus, the sheaf theories will not vary much, and hence we may assume firstly that G is semisimple.

Then, the proposition would follow from that every representation of $G^\vee$ is a subrepresentation of minuscule or quasi-minuscule representations. This is equivalent to

$$\bigoplus_{\substack{\lambda \text{ min.} \\ \text{or quasi-min.}}} V_\lambda = U$$

being a faithful representation of $G^\vee$. This is faithful because minuscule weights are in bijection with the center of $G^\vee$, so they detect the center of $G^\vee$ and we may assume $G^\vee$ is of adjoint type. Then, it suffices to prove that U is a faithful $\mathfrak{g}^\vee$-representation. Then, we may assume $\mathfrak{g}^\vee$ is in fact simple by tensoring the representations for different components. We consider the quasi-minuscule representation U' which has highest weight the highest short root. This gives a non-trivial representation of $\mathfrak{g}^\vee$, and since $\mathfrak{g}^\vee$ is simple, this representation is naturally faithful.

Thus the claim follows. $\square$

Proof. We first prove the easier lemma in the above, that is, Lemma 3.3.6. By highest weight formalism (say, in the paper[13]), it suffices to prove that the !- and $*$-pullback of $Z_{\mathcal{JW}}(V \otimes V')$ to each strata $i_w : \mathrm{Fl}^w \hookrightarrow \mathrm{Fl}$ is perverse (recall Fl^w is the I_0^--orbit). A priori, since we have $Z_{\mathcal{JW}}(V \otimes V')$ perverse, its $*$-pullback has perverse degree ≤ 0 and its !-pullback lies in perverse degree ≥ 0.

Recall Z is symmetric monoidal. Then, invoking the standard filtration on $Z_{\mathcal{JW}}(V)$, we have a filtration on the following

$$Z_{\mathcal{JW}}(V \otimes V') = \Delta_e *_I Z(V) *_I Z(V') = Z_{\mathcal{JW}}(V) *_I Z(V')$$

with subquotients $\Delta_w *_I Z(V')$. Then by centrality of $Z(V')$, we have

$$\Delta_w *_I Z(V') = \Delta_0 *_I j_{w!} *_I Z(V') = Z_{\mathcal{JW}}(V') *_I j_{w!}$$

Then, using the standard filtration on $Z_{\mathcal{IW}}(V')$, we obtain a filtration on the above graded pieces $Z_{\mathcal{IW}}(V') *_I j_{w!}$, itself with graded pieces $\Delta_u *_I j_{w!} = \Delta_e *_I (j_{u!} *_I j_{w!})$.

However, we claim that $j_{u!} *_I j_{v!}$ lies in the subcategory generated by $j_{w!}[n]$ under extensions, where $n \leq 0$. This is done by applying induction on the length of v, namely, for $\ell(v) = 1$ so v is a simple reflection, consider the exact sequence (which is also a triangle)

$$0 \to L_e \to j_{w!} \to L_w \to 0$$

For the case where $\ell(uv) = \ell(u) + 1$, by Lemma 2.1.5 we are done, and if $\ell(uv) = \ell(u) - 1$, we consider the two triangles

$$j_{u!} \to j_{u!} *_I j_{v!} \to j_{u!} *_I L_v, \qquad j_{u!}[-1] \to j_{u!} *_I L_v \to j_{uv!}$$

which successfully generates $j_{u!} *_I j_{v!}$. Thus, by applying the functor Av_Ψ, we have a double filtration of $Z_{\mathcal{IW}}(V \otimes V')$ with associated graded objects $Z_{\mathcal{IW}}(j_{u!} * j_{w!})$, which itself admits a filtration with associated graded objects $\Delta_v[n]$ with $[n] < 0$

Therefore, by the adjunction $i_{w!} \dashv i_w^!$, we obtain the !-restriction of each graded pieces of $Z_{\mathcal{IW}}(V \otimes V')$ has perverse degree ≤ 0. Dually, we have the $*$-restriction of the graded pieces of $Z_{\mathcal{IW}}(V \otimes V')$ has perverse degree ≥ 0. Hence we are done. $\square$

The more difficult lemma is Lemma 3.3.7, in particular, the proof of the quasi-minuscule part is very lengthy.

Lemma 3.3.8. *The statement of Lemma 3.3.7 holds for minuscule weights.*

Proof. Recall that we need to prove that $Z_{\mathcal{IW}}(V_\lambda)$ is tilting for λ a minuscule weight.

In the irreducible highest weight representation V_λ with λ a minuscule weight, the finite Weyl group action on the set of weights of V_λ is transitive. Then, following Lemma 3.2.5, we have Fl^λ open in the support of $Z_{\mathcal{IW}}(V_\lambda)$, and the stalk on the strata is given by $\Psi_\lambda^*(\mathbb{AS})[\ell(t_\lambda)]$. Hence, we have

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_\lambda, Z_{\mathcal{IW}}(V_\lambda)[n]) = 0$$

for $n > 0$. Dually, we have

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(Z_{\mathcal{IW}}(V_\lambda)[n], \nabla_\lambda) = 0$$

for $n > 0$. Now, to prove that $Z_{\mathcal{IW}}(V_\lambda)$ is tilting, we must prove that

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(\Delta_\mu, Z_{\mathcal{IW}}(V_\lambda)[n]\big) = 0, \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(Z_{\mathcal{IW}}(V_\lambda)[n], \nabla_\mu\big) = 0$$

for any $n > 0$ and all μ. Since the only possibly non-zero parts are the μ's that are weights of V_λ, hence $\mu \in W_f\lambda$, it suffices to prove that

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_\lambda, Z_{\mathcal{IW}}(V_\lambda)[n]) = \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_{w\lambda}, Z_{\mathcal{IW}}(V_\lambda)[n])$$

and its dual version. Then the result will follow from the first calculation.

For this assertion, we let v be the minimal length representative in $W_f t_{w\lambda}$, where $w \in W_f$. Then, recall that $wt_\lambda w^{-1} = t_{w\lambda}$. We actually have $v = t_\lambda u$, where $u \in W_f$ is of minimal length such that $uw\lambda = \lambda$. This element satisfies $\ell(v) = \ell(t_\lambda) - \ell(u)$. Hence, we have

$$\Delta_\lambda = \Delta_e *_I j_{t_\lambda !} = \Delta_e *_I j_{v!} *_I j_{u^{(-1)!}} = \Delta_{w\lambda} *_I j_{u^{-1}!}$$

Thus by invertibility of $j_{u^{-1}!}$, we may compute

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_{w\lambda}, Z_{\mathcal{IW}}(V_\lambda)[n]) = \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_{w\lambda} *_I j_{u^{-1}!}, Z_{\mathcal{IW}}(V_\lambda) *_I j_{u^{-1}!}[n])$$

Then, using Proposition 2.4.9, we have

$$Z_{\mathcal{IW}}(V_\lambda) *_I j_{u^{-1}!} = \Delta_e *_I Z(V_\lambda) *_I j_{u^{-1}!} = (\Delta_e *_I j_{u^{-1}!}) *_I Z(V_\lambda) = Z_{\mathcal{IW}}(V_\lambda)$$

Thus the claim follows.

Hence, we have $Z_{\mathcal{IW}}(V_\lambda)$ tilting for λ a minuscule weight. □

For the quasi-minuscule case, we need some geometric prerequisites.

Definition 3.3.9. We let $\mathcal{P}_I^0$ denote the quotient category

$$\mathcal{P}_I^0 = \mathbf{Perv}_I(\mathrm{Fl}) / \operatorname{Span}\{L_w | \ell(w) > 0\}$$

We denote Π^0 by the projection $\Pi^0 : \mathbf{Perv}_I(\mathrm{Fl}) \to \mathcal{P}_I^0$.

We define $Z^0 : \mathbf{Rep}\, G^\vee \to \mathcal{P}_I^0$ to be the composition

$$\mathbf{Rep}\, G^\vee \xrightarrow{Z} \mathbf{Perv}_I(\mathrm{Fl}) \twoheadrightarrow \mathcal{P}_I^0$$

and we let M_V^0 denote $\Pi^0(M_V)$, where M_V is the nilpotent endomorphism (recall its definition in Theorem 2.2.2).

Finally, we define $\tilde{\mathcal{P}}_I^0$ to be the full Abelian subcategory spanned by the subquotients of the image of Z^0. Hence, we have Z^0 factors through

$$\tilde{Z}^0 : \mathbf{Rep}\, G^\vee \to \tilde{\mathcal{P}}_I^0$$

Proposition 3.3.10. *The bifunctor*

$$\Pi^0({}^pH^0(- *_I -)) : \mathbf{Perv}_I(\mathrm{Fl}) \times \mathbf{Perv}_I(\mathrm{Fl}) \to \mathcal{P}_I^0$$

descends to a functor

$$- \star - : \mathcal{P}_I^0 \times \mathcal{P}_I^0 \to \mathcal{P}_I^0$$

exact on both sides, and admits natural associativity and unitality constraints. In other words, this gives a monoidal structure on $\mathcal{P}_I^0$.

Proof. For the exactness of the functor, it suffices to prove the following:

(1) For $\mathcal{F} \in \mathrm{Span}\{L_w | \ell(w) > 0\}$ and any $\mathcal{G} \in \mathbf{Perv}_I(\mathrm{Fl})$, we have ${}^pH^i(\mathcal{F} *_I \mathcal{G})$ and ${}^pH^i(\mathcal{G} *_I \mathcal{F})$ lying in the subcategory $\mathrm{Span}\{L_w | \ell(w)\} > 0$.

(2) $\Pi^0({}^pH^n(\mathcal{F} *_I \mathcal{G})) = 0$ for $n \neq 0$.

Then, we obtain the results for $- \star -$ directly.

Then, for the first claim, we only prove the claim for $\mathcal{F} *_I \mathcal{G}$, and the other is similar. By the long exact sequence of cohomology induced by exact triangles, we may assume $\mathcal{F}$ is of the form L_w where $\ell(w) > 0$. We choose a simple reflection s such that $\ell(sw) = \ell(w) - 1$. Then, as in the proof of Lemma 2.1.7, we pass to the category $\mathbf{Perv}_{I_s}(\mathrm{Fl})$, where I_s stands for the parahoric subgroup of LG. Thus, $\mathcal{F} *_I \mathcal{G} \in D_{I_s}(\mathrm{Fl})$, and again as in Lemma 2.1.7, any simple object that is I_s-equivariant cannot contain L_u for $\ell(u) = 0$ since its orbit cannot be I_s-equivariant. Thus, the first claim is proven.

For the second claim, again assume $\mathcal{F}, \mathcal{G} = L_{w_i}$ for some w_i by the long exact sequence. Then, if some w_i has $\ell(w_i) > 0$, we are done by the first claim, and otherwise if $\ell(w_i) = 0$, we have $L_{w_i} = j_{w_i!}$, and so

$$L_{w_1} *_I L_{w_2} = j_{w_1!} *_I j_{w_2!} = j_{w_1 w_2!} = L_{w_1 w_2}$$

□

Lemma 3.3.11. *$Z^0 : \mathbf{Rep}\, G^\vee \to \mathcal{P}_I^0$ is a central functor.*

Proof. Recall that Z and $\Pi^0({}^pH^0(-))$ are monoidal functors, hence so is Z^0. Then, the centrality isomorphism for Z gives

$$Z(V) *_I \mathcal{F} \xrightarrow{\simeq} \mathcal{F} *_I Z(V)$$

where both sides are Perverse sheaves by Theorem 2.2.1. Hence, applying Π^0 yields the centrality isomorphism for Z^0. $\square$

Corollary 3.3.12. *$\tilde{\mathcal{P}}_I^0$ inherits a monoidal structure from $\tilde{\mathcal{P}}_I^0 \subset \mathcal{P}_I^0$.*

Proof. This is because Z^0 is monoidal. Thus, let $\mathcal{F}_i$ be a subquotient of $Z^0(V_i)$, by bi-exactness of $- \star -$, $\mathcal{F}_1 \star \mathcal{F}_2$ is a subquotient of $Z^0(V_1) \star Z^0(V_2)$. $\square$

We will need the Tannakian information of the functor $\tilde{Z}^0$.

Construction 3.3.13. We construct a exact faithful monoidal functor

$$u : \tilde{\mathcal{P}}_I^0 \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$$

By a variant of Tannakian formalism, this functor provides an equivalence

$$\tilde{\mathcal{P}}_I^0 \xrightarrow{\simeq} \mathbf{coMod}_C$$

for some bialgebra C over $\overline{\mathbb{Q}}_\ell$. In addition, we will also show that the composition

$$\mathbf{Rep}\, G^\vee \xrightarrow{\tilde{Z}^0} \tilde{\mathcal{P}}_I^0 \xrightarrow{u} \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$$

is isomorphic to the forgetful functor.

Again, we begin with $\mathcal{O}(G^\vee) \in \operatorname{Ind} \mathbf{Rep}\, G^\vee$. Then, consider $\tilde{Z}^0(\mathcal{O}(G^\vee)) \in \operatorname{Ind} \tilde{\mathcal{P}}_I^0$. Recall $\operatorname{Ind} \tilde{\mathcal{P}}_I^0$ is an Abelian category. Moreover, since $\mathcal{O}(G^\vee)$ is a ring object and $\tilde{Z}^0$ is monoidal, we have $\tilde{Z}^0(\mathcal{O}(G^\vee)) \in \mathbf{Alg}\big(\operatorname{Ind} \tilde{\mathcal{P}}_I^0\big)$. Hence, we may consider its maximal left ideal object, which exists by Zorn's lemma (for any chain of left ideal objects, we may take the filtered colimit to obtain a maximal element in the chain).

Thus, we fix a left ideal object $\mathcal{J} \subset \tilde{Z}^0(\mathcal{O}(G^\vee))$. Then, writing $\mathcal{J}$ as a subquotient of $\tilde{Z}^0(V)$ (V may be an ind-representation), by the commutativity constraint

$$\tilde{Z}^0(\mathcal{O}(G^\vee)) \star \tilde{Z}^0(V) = \tilde{Z}^0(V) \star \tilde{Z}^0(\mathcal{O}(G^\vee))$$

we have $\mathcal{J}$ also a right ideal of $\tilde{Z}^0(\mathcal{O}(G^\vee))$. Hence, the quotient $\tilde{Z}^0(\mathcal{O}(G^\vee))/\mathcal{J}$ is a ring in $\operatorname{Ind} \tilde{\mathcal{P}}_I^0$, which we denote by $\mathcal{O}(H)$.

By construction, $\mathcal{O}(H)$ is simple, so $\operatorname{End}_{\mathbf{Mod}_{\mathcal{O}(H)}}(\mathcal{O}(H))$ is a division algebra. In particular, by the construction of ind-categories, it is at most of countable dimension. Thus by analysing the behavior of $1/(x-\lambda)$ for $\lambda \in \overline{\mathbb{Q}}_\ell$, we see that this division algebra is exactly $\overline{\mathbb{Q}}_\ell$ (this process is standard).

Now we construct the functor. We let

$$u(\mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}(H)}}(\mathcal{O}(H), \mathcal{O}(H) \star \mathcal{F}) : \tilde{\mathcal{P}}_I^0 \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$$

First, notice that $\mathcal{O}(H) \star \mathcal{F}$ is actually a finite free $\mathcal{O}(H)$-module. We first prove this for $\mathcal{F} = \tilde{Z}^0(V)$ for $V \in \mathbf{Rep}\, G^\vee$. Note that there is an isomorphism

$$\mathcal{O}(G^\vee) \otimes V \to \mathcal{O}(G^\vee) \otimes V^{\mathrm{triv}}$$

where V^{triv} is the trivial G-representation with the same underlying vector space as V. Then, applying $\tilde{Z}^0$ to both sides, and then quotienting out $\mathcal{I}$, we obtain

$$\mathcal{O}(H) \star \tilde{Z}^0(V) \simeq \mathcal{O}(H) \otimes_{\overline{\mathbb{Q}}_\ell} V^{\mathrm{triv}}$$

and since $\mathcal{O}(H)$ is simple, all subquotients of $\mathcal{O}(H)^{\oplus n}$ is simple, hence the claim. Moreover, this immediately implies $u\tilde{Z}^0$ is the forgetful functor.

Then, by the above, we obviously have u exact, and

$$\mathcal{O}(H) \star \mathcal{F} = \mathcal{O}(H) \otimes_{\overline{\mathbb{Q}}_\ell} u(\mathcal{F})$$

Hence

$$\mathcal{O}(H) \otimes_{\overline{\mathbb{Q}}_\ell} u(\mathcal{F} \star \mathcal{G}) = \mathcal{O}(H) \star \mathcal{F} \star \mathcal{G} = \mathcal{O}(H) \otimes_{\overline{\mathbb{Q}}_\ell} u(\mathcal{F}) \otimes u(\mathcal{G})$$

so u is monoidal.

Finally, for u to be faithful, it suffices to show that u kills no simple object. But this is because that the simple objects are $\star$-invertible, so u takes them to $\otimes$-invertible objects, hence not 0.

Proposition 3.3.14. *We have*

$$\tilde{\mathcal{P}}_I^0 \simeq \mathbf{Rep}\, H$$

for some $H \subset G^\vee$. Moreover, H is contained in $Z_{G^\vee}(m)$ (the centralizer for the adjoint action $G^\vee \circlearrowleft \mathfrak{g}^\vee$) for some $m \in \mathcal{N}^\vee \subset \mathfrak{g}^\vee$.

Proof. The previous construction gives a map of bialgebras over $\overline{\mathbb{Q}}_\ell$

$$\mathcal{O}(G^\vee) \to C$$

induced by $\tilde{Z}^0 : \mathbf{Rep}\, G^\vee \to \tilde{\mathcal{P}}_I^0$. Moreover, since every object right hand side is a subquotient of the image of $\tilde{Z}^0$, the map of bialgebras $\mathcal{O}(G^\vee) \to C$ is surjective, making

C a commutative bialgebra. Hence, C represents a closed submonoid of $G^\vee$. Thus, it represents a closed subgroup $H \subset G^\vee$ by algebraic geometry.

Then, for the second statement, note firstly that M_V is a nilpotent tensor derivation, hence represents a nilpotent element $m \in \mathfrak{g}^\vee$. Then, $H \subset Z(m)$ is equivalent to the condition that M_V gives a tensor derivation for the forgetful functor (this is because $\tilde{\mathcal{P}}_I^0$ is a subquotient of $\mathbf{Perv}_I(\mathrm{Fl})$)

$$\mathbf{Rep}\, G^\vee \to \mathbf{Rep}\, H \simeq \tilde{\mathcal{P}}_I^0$$

$\square$

Lemma 3.3.15. *The element m is regular.*

Proof. We must compute the dimension of $V^{m=0}$ for any $V \in \mathbf{Rep}\, G^\vee$, and then take in $V = \mathfrak{g}^\vee$. Recall that the forgetful functor $\mathbf{Rep}\, G^\vee \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$ is given by the composition

$$\mathbf{Rep}\, G^\vee \xrightarrow{\tilde{Z}^0} \tilde{\mathcal{P}}_I^0 \xrightarrow{u} \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$$

Then, recall that the dimension of the kernel of m is equal to the number of its Jordan blocks, so equal to $\dim \mathrm{gr}_0 V + \dim \mathrm{gr}_1 V$, where gr is the associated filtration of m on V. Note that $\dim u(L_w) = 1$ since L_w is invertible in $\tilde{\mathcal{P}}_I^0$, for u is monoidal. Thus, we have

$$\dim \mathrm{gr}_i V = \sum_{\omega \in \Omega \subset X_*(T)} [\mathrm{gr}_i^w Z(V) : L_\omega]$$

where the associated filtration of the nilpotent operator M_V on $Z(V)$ is the weight filtration, and we regard each graded piece a usual sheaf.

To compute the above summation, we invoke Proposition 2.5.8, and thus

$$[Z(V_\lambda)] = \sum_{\mu \in X_*(T)^+} \dim V_\lambda^{T^\vee = \mu} \cdot \theta_\mu$$

hence

$$[Z(V)] = \sum_{\mu \in X_*(T)^+} \dim V^{T^\vee = \mu} \cdot \theta_\mu$$

Then, consider the map $\eta : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ sending $H_w \mapsto (-v)^{\ell(w)}$. Thus, we have $\eta(B_\omega) = 1$ for all $\omega \in \Omega$ (or $\ell(\omega) = 0$), and $\eta(B_w) = 0$ for all $\ell(w) \neq 0$. The first identification is obvious, and for the second follows from $B_s = H_s + v$ in the case of s a simple reflection, so by induction on the length, we apply Proposition 3.3.10, and see that it holds for all B_w for $\ell(w) > 0$. Moreover, we have $\eta(\theta_\lambda) = v^{\langle \lambda, 2\rho \rangle}$ by letting $\lambda = \mu - \nu$ for $\mu, \nu \in X_*(T)^+$.

Hence, applying η we have

$$\sum_{i \in \mathbb{Z}} \sum_{\omega \in \Omega} [\mathrm{gr}_i^w(Z(V)) : L_w] v^i = \sum_{\mu \in X_*(T)} \dim V^{T^\vee = \mu} \cdot v^{\langle \mu, 2\rho \rangle}$$

comparing both sides, we have

$$\dim V^{m=0} = \sum_{i=0,1} \sum_{\omega \in \Omega} [\mathrm{gr}_i^w(Z(V)) : L_w] = \sum_{\substack{\mu \in X_*(T) \\ \langle \mu, 2\rho \rangle \in \{0,1\}}} \dim V^{T^\vee = \mu}$$

Finally, taking in $V = \mathfrak{g}^\vee$, we have

$$\dim \mathfrak{Z}_{\mathfrak{g}^\vee}(m) = \operatorname{rank} T^\vee$$

and hence m is regular. $\square$

Lemma 3.3.16. *We have inequalities*

$$\dim\Big(\mathrm{Hom}_{\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V))\Big) \leq \dim V^{m=0}$$

$$\dim\Big(\mathrm{Hom}_{\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})}(Z_{\mathcal{IW}}(V), \nabla_e)\Big) \leq \dim V^{m=0}$$

Proof. We only prove the first inequality, since the second is analogous.

Recall that we have a fully faithful embedding ${}^f\mathcal{P}_I \hookrightarrow \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$. Hence, we have

$$\mathrm{Hom}_{\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)) = \mathrm{Hom}_{{}^f\mathcal{P}_I}\big({}^f\delta_e, {}^fZ(V)\big)$$

Then, further applying Π^0 gives a map

$$\mathrm{Hom}_{{}^f\mathcal{P}_I}\big({}^f\delta_e, {}^fZ(V)\big) \to \mathrm{Hom}_{\mathcal{P}_I^0}\Big(Z^0\big(\overline{\mathbb{Q}}_\ell\big), Z^0(V)\Big)$$

This is injective because ${}^f\delta_e$ is a simple object not killed by Π^0.

Finally, since we have $M^0_{\overline{\mathbb{Q}}_\ell} = 0$, using that M^0_V is a natural transformation we have

$$\mathrm{Hom}_{\mathcal{P}_I^0}\Big(Z^0\big(\overline{\mathbb{Q}}_\ell\big), Z^0(V)\Big) = \mathrm{Hom}_{\mathcal{P}_I^0}\Big(Z^0\big(\overline{\mathbb{Q}}_\ell\big), \ker\big(M^0_V\big)\Big)$$

Finally, applying the faithful functor u we have

$$\mathrm{Hom}_{\mathcal{P}_I^0}\Big(Z^0\big(\overline{\mathbb{Q}}_\ell\big), \ker\big(M^0_V\big)\Big) \hookrightarrow V^{m=0}$$

hence the lemma is proven. $\square$

We can finally prove the quasi-minuscule case.

Lemma 3.3.17. *The statement of Lemma 3.3.7 holds for quasi-minuscule weights.*

Proof. Now, we must prove $Z_{\mathcal{IW}}(V_\lambda)$ is tilting for λ a quasi-minuscule dominant coweight.

We have $Z_{\mathcal{IW}}(V_\lambda)$ supported on

$$\overline{\mathrm{Fl}^\lambda} = \bigcup_{\mu \in W_f \cdot \lambda \cup 0} \mathrm{Fl}^\mu$$

By the same method as in the proof of Lemma 3.3.8, we obtain

$$\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(\Delta_\mu, Z_{\mathcal{IW}}(V_\lambda)[n]\big) = 0, \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(Z_{\mathcal{IW}}(V_\lambda)[n], \nabla_\mu\big) = 0$$

for all $n > 0$, $\mu \in W_f \cdot \lambda$ It now suffices to prove the case where $\mu = 0$. Consider the closed embedding $i : \overline{\mathrm{Fl}^0} \to \mathrm{Fl}$, and let j be its complement open. Then, consider the triangles

$$j_! j^* Z_{\mathcal{IW}}(V) \to Z_{\mathcal{IW}}(V) \to i_* i^* Z_{\mathcal{IW}}(V)$$

$$i_* i^! Z_{\mathcal{IW}}(V) \to Z_{\mathcal{IW}}(V) \to j_* j^* Z_{\mathcal{IW}}(V)$$

Since we have the tiltingness of $Z_{\mathcal{IW}}(V)$ outside $\overline{\mathrm{Fl}^0}$, we have $j_! j^* Z_{\mathcal{IW}}(V)$ and $j_* j^* Z_{\mathcal{IW}}(V)$ perverse, hence $i^* Z_{\mathcal{IW}}$ is concentrated in perverse degree 0 and −1, while $i^! Z_{\mathcal{IW}}$ is concentrated in perverse degree 0 and 1. Then, by the proof of Lemma 3.3.15, we have

$$\dim V^{m=0} = \sum_{\substack{\mu \in X_*(T) \\ \langle \mu, 2\rho \rangle \in \{0,1\}}} \dim V^{T^\vee = \mu}$$

Since in our case we are treating with quasi-minuscule $V = V_\lambda$, all non-zero weights appearing in V_λ are $W_f \cdot \lambda$, and since a quasi-minuscule dominant coweight must be a coroot, we have all non-zero weights appearing in V_λ are actually coroots of G (hence roots of $G^\vee$), so the only possible μ with $\langle \mu, 2\rho \rangle \in \{0, 1\}$ is 0.

Hence, by Lemma 3.3.16, we have

$$\dim \mathrm{Hom}_{\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)) \leq \dim V^{T^\vee = 0}$$

Then by the proof of Lemma 3.3.5, we have the identity

$$\sum_{i\geq 0}(-1)^i \dim \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)[i]) = \dim V^{T^\vee=0}$$

But then, using that $i^! Z_{\mathcal{IW}}(V)$ is concentrated in perverse degrees 0 and 1, we obtain

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V))\\ \leq\ &\dim \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)) - \dim \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)[1])\end{aligned}$$

so the latter term must be 0. Hence, $i^! Z_{\mathcal{IW}}(V)$ is perverse. Dually, we have $i^* Z_{\mathcal{IW}}(V)$ perverse, and thus $Z_{\mathcal{IW}}(V)$ is tilting. □

3.4. The proof of the final equivalence

Now we may finish up the proof of Theorem 2.4.2.

The strategy of the proof is to show full faithfulness on certain generators and then essential surjectivity.

Proposition 3.4.1. *The category $D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)$ is generated as a triangulated category by the following classes of objects:*

(1) The line bundles $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ for $\lambda \in X_*(T)$.

(2) The vector bundles $V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ for $V \in \mathbf{Rep}\, G^\vee$ or $\lambda \in X_*(T)^+$.

Here, recall $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ can be described as the pullback of the bundle $\mathcal{O}_{\mathcal{B}^\vee}(\lambda)$ from the projection $\widetilde{\mathcal{N}}^\vee \twoheadrightarrow \mathcal{B}^\vee$, and $\mathcal{O}_{\mathcal{B}^\vee}(\lambda)$ is the line bundle obtained from Borel-Weil-Bott theorem.

Proof.

(1) For these generators, recall

$$\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) = \mathrm{Coh}^{G^\vee}\left(G^\vee \times^{B^\vee} \mathfrak{n}^\vee\right) = \mathrm{Coh}^{B^\vee}(\mathfrak{n}^\vee)$$

with the identification induced by the restriction along the inclusion

$$\{1\} \times \mathfrak{n}^\vee \hookrightarrow \widetilde{\mathcal{N}}^\vee$$

Thus, the modules $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ corresponds to the $B^\vee$-equivariant $\mathcal{O}(\mathfrak{n}^\vee)$-modules $k_{B^\vee}(\lambda) \otimes \mathcal{O}(\mathfrak{n}^\vee)$. Note that they generate the category $V \otimes \mathcal{O}(\mathfrak{n}^\vee)$ for $V \in \mathbf{Rep}\, B^\vee$, so it suffices to prove the latter generate the triangulated category.

Consider the cocharacter $2\rho : \mathbb{G}_m \to T^\vee$. Using restriction along this map, we regard any $T^\vee$- so $B^\vee$-equivariant module as a graded module. Then, consider the projective resolution of an $B^\vee$-equivariant module M

$$\cdots \to P^{-1} \to P^0 \to M \to 0$$

where each P^j is of the form $V \otimes \mathcal{O}(\mathfrak{n}^\vee)$ for some $B^\vee$-representation V. Thus, this gives rise to a sequence of graded $\mathcal{O}(\mathfrak{n}^\vee)$-modules. Since $\mathfrak{n}^\vee$ is a vector space, let m denote its dimension. By Hilbert's syzygy theorem on dimensions of polynomial rings, $\ker(P^m \to P^{m-1})$ is finite free as a graded $\mathcal{O}(\mathfrak{n}^\vee)$-module (apply the syzygy theorem to each grading).

Now, it suffices to show that any $B^\vee$-equivariant $\mathcal{O}(\mathfrak{n}^\vee)$-module free as a $\mathcal{O}(\mathfrak{n}^\vee)$ -module admits a $B^\vee$-filtration with subquotients of the form $V \otimes \mathcal{O}(\mathfrak{n}^\vee)$ for $V \in \mathbf{Rep}\, B^\vee$. This is obvious by induction on the dimension.

(2) We use these generators to reach the generators in (1). For $\lambda \in X_*(T)^+$, consider the highest weight line $B_\lambda : V_\lambda \otimes \mathcal{O} \to \mathcal{O}(\lambda)$ as in Construction 3.2.4. This gives a surjective morphism of modules $V_\lambda \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee} \to \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$. Taking the Koszul complex of this surjection, we obtain an exact sequence (we set $d = \dim V_\lambda$)

$$0 \to \wedge^d V_\lambda \to \wedge^{d-1} V_\lambda \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda) \to \cdots \to \wedge^1 V_\lambda \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}((d-1)\lambda) \to \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(d\lambda) \to 0$$

Then, for any $\mu \in X_*(T)$, we pick λ such that $\mu + \lambda$ is dominant. Hence, tensoring the above sequence with $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\mu) \otimes \left(\wedge^d V_\lambda\right)^*$ gives the generation of $\mathcal{O}_{\widetilde{\mathcal{N}}}(\mu)$. $\square$

We first make more use of the regularity of m.

Construction 3.4.2. Recall that the regular nilpotent orbit $\mathcal{O}_r^\vee \subset \mathcal{N}^\vee$ has its preimage $\tilde{\mathcal{O}}_r^\vee \subset \widetilde{\mathcal{N}}^\vee$, and the Springer resolution restricts to an isomorphism $\tilde{\mathcal{O}}_r^\vee \to \mathcal{O}_r^\vee$. Also, by the definition of orbits, $G^\vee / Z_{G^\vee}(m) \to \mathcal{O}^\vee$ is an isomorphism. Thus, we have an equivalence of categories

$$\mathrm{Coh}^{G^\vee}\left(\tilde{\mathcal{O}}_r^\vee\right) \simeq \mathbf{Rep}(Z_{G^\vee}(m))$$

Then, let $\mathrm{Coh}^{G^\vee}_{\mathrm{fr}}\left(\widetilde{\mathcal{N}}^\vee\right)$ denote the full subcategory of $\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)$ consisting of objects of the form $V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$, we have a functor

$$\mathrm{Coh}^{G^\vee}_{\mathrm{fr}}\left(\widetilde{\mathcal{N}}\right) \to \mathrm{Coh}^{G^\vee}\left(\tilde{\mathcal{O}}_r^\vee\right) \to \mathbf{Rep}(Z_{G^\vee}(m)) \to \mathbf{Rep}\, H$$

where the first functor is restriction, the second is an equivalence, and the third is induced by the restriction along the inclusion shown in Proposition 3.3.14. Moreover, the composition of the first two functors can be identified to the restriction at $\tilde{m} \to m$, where $\tilde{m}$ is the unique preimage of m.

On the other hand, we have a detour given by

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \xrightarrow{\Phi} \mathbf{Perv}_I(\mathrm{Fl}) \xrightarrow{\Pi^0} \tilde{\mathcal{P}}_I^0 \xrightarrow{\simeq} \mathbf{Rep}\, H$$

These two functors admit a natural monoidal structure. We claim that these two functors are actually naturally isomorphic as monoidal functors.

Recall we have shown in Proposition 3.2.9 and Construction 3.3.13 that the two functors coincide as monoidal functors after postcomposing with the forgetful functor $\mathbf{Rep}\, H \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$. Thus, it suffices to show that the maps

$$\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}\left(V_1 \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V_2 \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right) \to \mathrm{Hom}_{\mathbf{Rep}\, H}(V_1, V_2)$$

induced by the functors coincide.

By definition, the left hand side can be identified with $\left(V_1^* \otimes V_2 \otimes \mathcal{O}\left(\widetilde{\mathcal{N}}^\vee\right)\right)^{G^\vee}$. Then, recall that $\mathcal{O}\left(\widetilde{\mathcal{N}}^\vee\right) = \mathcal{O}(\mathcal{N}^\vee)$, so we have a surjection $\mathcal{O}(\mathfrak{g}^\vee) \twoheadrightarrow \mathcal{O}\left(\tilde{\mathcal{N}}^\vee\right) = \mathcal{O}\left(\tilde{\mathcal{N}}^\vee\right)$. Thus, the two functors coincide iff their precomposition with the pullback

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee}(\mathfrak{g}^\vee) \to \mathrm{Coh}_{\mathrm{fr}}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \rightrightarrows \mathbf{Rep}\, H$$

coincide. However, their further precomposition to

$$\mathbf{Rep}\, G^\vee \to \mathrm{Coh}_{\mathrm{fr}}^{G^\vee}(\mathfrak{g}^\vee) \to \mathrm{Coh}_{\mathrm{fr}}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \rightrightarrows \mathbf{Rep}\, H$$

coincide, and so an extension of this functor to functors originating from $\mathrm{Coh}_{\mathrm{fr}}^{G^\vee}(\mathfrak{g}^\vee)$ is equivalent to an tensor derivation of the map $\mathbf{Rep}\, G^\vee \to \mathbf{Rep}\, H$. However, it is by construction that both tensor derivation is given by M.

Lemma 3.4.3. *For any $V \in \mathbf{Rep}\, G^\vee$, the morphism*

$$\mathrm{Hom}_{D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right) \to \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\left(\Phi_{\mathcal{IW}}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right), \Phi_{\mathcal{IW}}\left(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right)\right)$$

is injective.

Proof. By definition, we have $\Phi_{\mathcal{IW}}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right) = \Delta_e$, and $\Phi_{\mathcal{IW}}\left(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right) = Z_{\mathcal{IW}}(V)$. Then, using ${}^f\mathcal{P}_I \hookrightarrow \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$ is fully faithful, it suffices to show

$$\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V\otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big) \to \mathrm{Hom}_{{}^f\mathcal{P}_I}\big({}^f\delta_e, {}^fZ(V)\big)$$

We may further apply Π^0 and show

$$\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V\otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big) \to \mathrm{Hom}_{\tilde{\mathcal{P}}_I^0}\big(\delta_e^0, \tilde{Z}^0(V)\big) = \mathrm{Hom}_{\mathbf{Rep}\,H}\big(\overline{\mathbb{Q}}_\ell, V\big)$$

is injective.

By the previous construction, we can identify this functor with the restriction at $\tilde{m}$. But note that the left hand side is identified with $\big(V\otimes\mathcal{O}\big(\widetilde{\mathcal{N}}^\vee\big)\big)^{G^\vee}$. Since $\tilde{\mathcal{O}}_r^\vee \subset \widetilde{\mathcal{N}}^\vee$ has complement of codim 2, we have $\mathcal{O}(\mathcal{N}^\vee) = \mathcal{O}(\mathcal{O}_r^\vee)$. Then, since $\mathcal{O}_r^\vee = G^\vee/Z_{G^\vee}(m)$, we have

$$(V\otimes\mathcal{O}(\mathcal{O}_r^\vee))^{G^\vee} = \Big(V\otimes \mathrm{ind}_{Z_{G^\vee}(m)}^{G^\vee}\Big)^{G^\vee} = V^{Z_{G^\vee}(m)}$$

which clearly embeds into V^H since $H\subset Z_{G^\vee}(m)$. $\square$

Proposition 3.4.4. *For any $V\in\mathbf{Rep}\,G^\vee$, $\lambda\in X_*(T)^+$, and $n\in\mathbb{Z}$, the morphism*

$$\mathrm{Hom}_{D\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)[n]\big) \to \mathrm{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\big(\Phi_{\mathcal{JW}}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big), \Phi_{\mathcal{JW}}\big(V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\big)[n]\big)$$

is injective.

Proof. Firstly we must treat the case where $n>0$. By the definition of equivariant derived categories, we may identify the Hom group on the left as follows.

$$\begin{aligned}
&\mathrm{Hom}_{D\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)[n]\big)\\
&= \mathrm{Hom}^{G^\vee}_{D\mathrm{Coh}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)[n]\big)\\
&= \Big(V\otimes \mathrm{Hom}_{D\mathrm{Coh}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)[n]\big)\Big)^{G^\vee}\\
&= \Big(V\otimes H^n\big(\widetilde{\mathcal{N}}^\vee, \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\big)\Big)^{G^\vee}
\end{aligned}$$

But by the paper[14], such cohomologies vanish unless $n=0$. Thus we may assume $n=0$, since the cases in the rest is trivially true.

Then, we find some V such that $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\hookrightarrow V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$. For $\alpha\in R_+^\vee$, consider the exact sequence of vector bundles

$$G^\vee\times^{B^\vee}\mathfrak{n}^\vee_{(\alpha)} \to G^\vee\times^{B^\vee}\mathfrak{n}^\vee \to G^\vee\times^{B^\vee}\mathfrak{g}^\vee_\alpha$$

where $\mathfrak{g}_\alpha^\vee$ is the root space, $\mathfrak{n}_\alpha^\vee$ is the vector subspace of $\mathfrak{n}^\vee$ spanned by positive roots except α. The above exact sequence can be explicitly written out as

$$\widetilde{\mathcal{N}}_{(\alpha)}^\vee \to \widetilde{\mathcal{N}}^\vee \to \mathcal{O}_{\mathcal{B}^\vee}(-\alpha)$$

This allows us to express the ideal sheaf associated to $\widetilde{\mathcal{N}}_{(\alpha)}^\vee$ as $p^*\mathcal{O}_{\mathcal{B}^\vee}(\alpha) = \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\alpha)$. Anyway, we obtain an inclusion $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\alpha) \hookrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$. Thus by tensoring, this holds for general $\alpha \in \mathbb{Z}R_+^\vee$. Then for general λ, we choose some sufficiently large $\alpha \in \mathbb{Z}R_+^\vee$ such that $\lambda - \alpha \in -X_*(T)^+$. This gives an embedding $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda) \hookrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda - \mu)$. Then, since $\lambda - \mu$ is antidominant, we have an embedding $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda - \mu) \hookrightarrow V_{w_0(\lambda-\mu)} \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$. Thus, the composition of the embeddings gives the result.

Finally, we have

$$\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\big) \hookrightarrow \mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes V' \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big)$$

for some V'. By Lemma 3.4.3, we have the embedding

$$\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes V' \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big) \hookrightarrow \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(\Phi_{\mathcal{IW}}(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}), \Phi_{\mathcal{IW}}(V \otimes V' \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee})\big)$$

Hence, we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\big) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, V \otimes V' \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\big) \arrow[d, hook] \\
\mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(\Phi_{\mathcal{IW}}(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}), \Phi_{\mathcal{IW}}(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda))\big) \arrow[r] & \mathrm{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}\big(\Phi_{\mathcal{IW}}(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}), \Phi_{\mathcal{IW}}(V \otimes V' \otimes \mathcal{O}_{\widetilde{\mathcal{N}}^\vee})\big)
\end{tikzcd}$$

thus the left map must be injective. □

The above proposition essentially proves full faithfulness on the generators. We then move to essential surjectivity.

Proposition 3.4.5. *The objects* $\mathrm{Av}_\Psi(J_\lambda)$ *for* $\lambda \in X_*(T)$ *generates* $D_{\mathcal{IW}}(\mathrm{Fl})$ *as a triangulated category.*

Proof. Recall $[j_{w!}] = [j_{w*}]$, so $[J_\lambda] = \left[j_{t_\lambda !}\right] \in K^0(D_I(\mathrm{Fl}))$. Thus, by Proposition 2.4.9, we have $[\mathrm{Av}_\Psi(J_\lambda)] = [\Delta_\lambda] \in K^0(D_{\mathcal{IW}}(\mathrm{Fl}))$.

To show the generation, it suffices to show that $\mathrm{Av}_\Psi(J_\lambda)$ is supported on $\overline{\mathrm{Fl}^\lambda}$ with restriction to Fl^λ of rank 1. Thus, since $\mathrm{Av}_\Psi(J_\lambda)$ is perverse, its Euler characteristic of the stalk

$$\sum_{i\in\mathbb{Z}}(-1)^i H^{i+\ell(w)}\big(\mathrm{Fl}^\mu, i_\mu^! \operatorname{Av}_\Psi(J_\lambda)\big)$$

is concentrated in one degree, so equal to the dimension of the stalk. Here w is the minimal length element in $W_f\cdot t_\lambda$.

In particular, by the equality $[\operatorname{Av}_\Psi(J_\lambda)]=[\Delta_\lambda]$ and Euler characteristics factors through K-groups, this Euler characteristic is the same as the Euler characteristic

$$\sum_{i\in\mathbb{Z}}(-1)^i H^{i+\ell(w)}\big(\mathrm{Fl}^\mu, i_\mu^! \Delta_\lambda\big)$$

But the support and open stalk of Δ_λ is obvious.

Thus, by an induction on lengths, we can show that these objects generate the simple perverse sheaves, proving the generation. □

Proof. Now we will prove Theorem 2.4.1. Namely, we shall prove that

$$\Phi_{\mathcal{IW}}: D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)\to D_{\mathcal{IW}}(\mathrm{Fl})$$

is an equivalence of categories.

First, essential surjectivity follows from Proposition 3.4.5 and full faithfulness, since by definition the image of $\Phi_{\mathcal{IW}}$ contains the image of $\operatorname{Av}_\Psi\circ\Phi_0$, and $\Phi_0(V_0\otimes k_\lambda)=J_\lambda$, so $\operatorname{Av}_\Psi(J_\lambda)$ is contained in the image of $\Phi_{\mathcal{IW}}$.

Then we show full faithfulness. We temporarily denote the following map by f, and we will show

$$f:\operatorname{Hom}_{D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}(\mathcal{F},\mathcal{G})\to\operatorname{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Phi_{\mathcal{IW}}(\mathcal{F}),\Phi_{\mathcal{IW}}(\mathcal{G}))$$

is an isomorphism.

We first show the injectivity for the case $\mathcal{F}=\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$ and $\mathcal{G}=V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)[n]$, where $V\in\mathbf{Rep}\,G^\vee$ and $\lambda\in X_*(T)^+$. Then, this map is an injection by Proposition 3.4.4. We claim the source and target of f are finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces of the same dimension, which will prove that f is an isomorphism.

For the right hand side, it can be identified with

$$\begin{aligned}&\operatorname{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e, Z_{\mathcal{IW}}(V)*_I J_\lambda[n])\\ =&\operatorname{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_e *_I J_{-\lambda}, Z_{\mathcal{IW}}(V)[n])\\ =&\operatorname{Hom}_{D_{\mathcal{IW}}(\mathrm{Fl})}(\Delta_{-\lambda}, Z_{\mathcal{IW}}(V)[n])\end{aligned}$$

Thus by Proposition 3.3.4, this is zero unless $n=0$, and for $n=0$ the dimension is $\dim V^{T^\vee=-\lambda}$.

For the left hand side, in the proof of Proposition 3.4.4 we have shown that it vanishes unless $n=0$, and for $n=0$, it can be identified with

$$\left(V\otimes H^0\left(\widetilde{\mathcal{N}}^\vee,\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\right)\right)^{G^\vee}$$

So its dimension can be computed by the simple factors, namely, it is of dimension

$$\sum_{\nu\in X_*(T)^+}[V^*:V_\nu]\cdot\left[H^0\left(\widetilde{\mathcal{N}}^\vee,\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\right):V_\nu\right]$$

Again by the paper[14], the second factor is $\dim V_\nu^{T^\vee=\lambda}$, so

$$\sum_{\nu\in X_*(T)^+}[V^*:V_\nu]\cdot\dim V_\nu^{T^\vee=\lambda}=\dim\left(V^*\right)^{T^\vee=\lambda}=\dim V^{T^\vee=-\lambda}$$

Thus this part is proven.

Now, since $V\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ generates the whole triangulated category by Proposition 3.4.1, we have finished the proof that f is isomorphic for $\mathcal{F}=\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$ and $\mathcal{G}$ any sheaf. Now, take $\mathcal{F}=\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ for all possible $\lambda\in X_*(T)$, we have

$$\operatorname{Hom}_{D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda),\mathcal{G}\right)=\operatorname{Hom}_{D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee},\mathcal{G}\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-\lambda)\right)$$

and

$$\begin{aligned}&\operatorname{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\left(\Phi_{\mathcal{JW}}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)\right),\Phi_{\mathcal{JW}}(\mathcal{G})\right)\\=\;&\operatorname{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}(\Delta_e*_I J_\lambda,\Phi_{\mathcal{JW}}(\mathcal{G}))\\=\;&\operatorname{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}(\Delta_e,\Phi_{\mathcal{JW}}(\mathcal{G})*_I J_{-\lambda})\\=\;&\operatorname{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\left(\Phi_{\mathcal{JW}}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right),\Phi_{\mathcal{JW}}\left(\mathcal{G}\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-\lambda)\right)\right)\end{aligned}$$

by the monoidality of $\Phi_{\mathcal{JW}}$. Thus, since we have an isomorphism

$$\operatorname{Hom}_{D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right)}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee},\mathcal{G}\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-\lambda)\right)=\operatorname{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}\left(\Phi_{\mathcal{JW}}\left(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\right),\Phi_{\mathcal{JW}}\left(\mathcal{G}\otimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-\lambda)\right)\right)$$

by the case for $\mathcal{F}=\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$, the case where $\mathcal{F}=\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ and arbitrary $\mathcal{G}$ follows. Then, since $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ also generates the triangulated category by Proposition 3.4.1, we have finished the proof of full faithfulness.

Thus, $\Phi_{\mathcal{JW}}$ is an equivalence of categories! □

Proof. We now prove Theorem 2.4.2, namely,

$$\mathrm{Av}_\Psi : {}^f\mathcal{P}_I \xrightarrow{\simeq} \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$$

Recall we have proven in Proposition 2.4.10 that Av_Ψ is fully faithful. To prove essential surjectivity, note by Theorem 2.4.1 that every $\mathcal{F} \in \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$, there is some $\mathcal{G} \in D_I(\mathrm{Fl})$ such that $\mathrm{Av}_\Psi(\mathcal{G}) = \mathcal{F}$. Thus,

$$\mathcal{F} = \mathrm{Av}_\Psi({}^pH^0(\mathcal{G})) = \mathrm{Av}_\Psi\left({}^f({}^pH^0(\mathcal{G}))\right)$$

so essential surjectivity follows. $\square$

Chapter 4 The graded setting

4.1. Remarks on graded sheaves

In this chapter, we will make use of the graded sheaves introduced in the paper[8]. We summarize here the main properties we will use on graded sheaves.

Definition 4.1.1 (Definition 2.4.2[8]). Let X be a stack over k with a model X_0 over $\mathbb{F}_q$. We let $D_{\text{mixed}}(X_0)^{\text{ren}}$ denote $\text{Ind}(D_{\text{mixed}}(X_0))$. Then, we let $D_{\text{gr}}(X_0)^{\text{ren}}$ denote the Lurie tensor product

$$D_{\text{gr}}(X_0)^{\text{ren}} = D_{\text{mixed}}(X_0)^{\text{ren}} \otimes_{D_{\text{mixed}}(\operatorname{Spec}\mathbb{F}_q)^{\text{ren}}} \mathbf{Vect}^{\text{gr}}$$

where $D_{\text{mixed}}(\operatorname{Spec}\mathbb{F}_q)^{\text{ren}} \to D_{\text{mixed}}(X_0)^{\text{ren}}$ is given by the modules structure, and the functor $D_{\text{mixed}}(\operatorname{Spec}\mathbb{F}_q)^{\text{ren}} \to \mathbf{Vect}^{\text{gr}}$ is the functor sending a constructible sheaf to the associated graded vector space of its stalk. Here the stalk has the weight filtration.

We define $D_{\text{gr}}(X_0)$ to be the compact objects in $D_{\text{gr}}(X_0)^{\text{ren}}$.

Theorem 4.1.2 (Proposition 2.7.9). *Let $X_1 = X_0 \otimes \mathbb{F}_{q^n}$. There is a monoidal equivalence*

$$D_{\text{gr}}(X_0)^{\text{ren}} \simeq D_{\text{gr}}(X_1)^{\text{ren}}$$

Theorem 4.1.3 (Theorem 2.7.12). *We can attach to each stack X over k with a model X_0 over $\mathbb{F}_q$ a presentably symmetric monoidal stable ∞-category $D_{\text{gr}}(X)^{\text{ren}} := D_{\text{gr}}(X_0)^{\text{ren}}$, well-defined by the previous theorem. Moreover, for nice enough morphisms $f : X \to Y$, there is a theory of $f_! \dashv f^!$ and $f^* \dashv f_*$ between the categories $D_{\text{gr}}(X)^{\text{ren}}$ and $D_{\text{gr}}(Y)^{\text{ren}}$.*

Theorem 4.1.4 (Lemma 2.4.3, 2.4.7). *There is a symmetric monoidal continuous and compact preserving functor*

$$\text{gr} : D_{\text{mixed}}(X)^{\text{ren}} \to D_{\text{gr}}(X)^{\text{ren}}$$

Moreover, the image of gr *of compact objects compactly generates $D_{\text{gr}}(X)^{\text{ren}}$.*

Theorem 4.1.5 (Remark 2.7.17). *We let* $\langle n \rangle$ *denote the grade shift functor of* $D_{\mathrm{gr}}(X)$. *Then, there is a functor* $\mathrm{oblv}_{\mathrm{gr}} : D_{\mathrm{gr}}(X) \to D(X)$ *satisfying*

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}\langle n \rangle) = \mathrm{Hom}\big(\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}), \mathrm{oblv}_{\mathrm{gr}}(\mathcal{G})\big)$$

Theorem 4.1.6 (Theorem 3.4.8, Proposition 3.5.1). *The category* $D_{\mathrm{gr}}(X)$ *is equipped with a weight structure and a t-structure, such that the t-structure is transversal to the weight structure. We will call this t-structure the perverse t-structure. Moreover, the functors* $\mathrm{gr} : D_{\mathrm{mixed}}(X) \to D_{\mathrm{gr}}(X)$ *and* $\mathrm{oblv}_{\mathrm{gr}} : D_{\mathrm{gr}}(X) \to D(X)$ *preserve and reflect the perverse t-structures, and are compatible with the six functors.*

Theorem 4.1.7 (Proposition 3.5.6). *The category* $\mathbf{Perv}_{\mathrm{gr}}(X)$ *is of finite length. An object* $\mathcal{F} \in \mathbf{Perv}_{\mathrm{gr}}(X)$ *is simple iff* $\mathrm{oblv}(\mathcal{F}) \in \mathbf{Perv}(X)$ *is simple, and is necessarily pure of some weight.*

Definition 4.1.8. We define $D_{I,\mathrm{gr}}(\mathrm{Fl})$ to be the category of graded sheaves over the stack $I \backslash LG/I$. More precisely, we need to go through the process of pro-groups acting on ind-schemes[9]. First, we pick a split model G_0 of G over $\mathbb{F}_q$. Hence, we can choose a Borel defined over $\mathbb{F}_q$. The loop group and the Iwahori can also be defined over $\mathbb{F}_q$. Then, we may define

$$D_{I_0,\mathrm{mixed}}(\mathrm{Fl}_0) := \operatorname*{colim}_{w \in W^{\mathrm{ext}}} D_{(I_0)_k,\mathrm{mixed}}\Big(\big(\overline{\mathrm{Fl}_{0,w}}\big)\Big)$$

Here, $(I_0)_k$ is a finite type quotient of I_0 through which the action of I_0 on $\overline{\mathrm{Fl}_{0,w}}$ factors. Based on this, we may define the corresponding renormalized versions. Then, we follow the above formalism and define the associated graded category and its compact objects. Thus we obtain the category $D_{I,\mathrm{gr}}(\mathrm{Fl})$. In particular, the inclusions of strata $j_w : I\backslash \mathrm{Fl}_w \hookrightarrow I\backslash \mathrm{Fl}$ are nice morphisms in the sense of Theorem 4.1.3, and we have access to the !- and $*$-pushforwards and pullbacks for these morphisms.

The category $D_{\mathcal{IW},\mathrm{gr}}(\mathrm{Fl})$ is defined through a likewise process.

Definition 4.1.9. We define the graded versions of standard, costandard, and IC sheaves. We define for each $w \in W$ the following graded sheaves

$$j^{\mathrm{gr}}_{w!} = j_{w!}\overline{\mathbb{Q}}_\ell[\ell(w)]\langle \ell(w)\rangle, j^{\mathrm{gr}}_{w*} = j_{w*}\overline{\mathbb{Q}}_\ell[\ell(w)]\langle \ell(w)\rangle$$

and we define the corresponding simple IC object by

$$L_w^{\mathrm{gr}} = j_{w!*}\overline{\mathbb{Q}}_\ell[\ell(w)]\langle\ell(w)\rangle$$

By definition, we have $j_{w!}^{\mathrm{gr}} = \mathrm{gr}(j_{w!}^m)$ and the likewise statement for costandards and simple objects.

Construction 4.1.10. By the general formalism introduced in Construction 2.1.2, we may define the convolution product of graded sheaves. In particular, this convolution product is compatible with gr and $\mathrm{oblv}_{\mathrm{gr}}$, since the two functors preserve the six-functor formalism.

Corollary 4.1.11. *The lemma Lemma 2.1.5 holds with $j_{w!}$ replaced by $j_{w!}^{\mathrm{gr}}$, and j_{w*} replaced by j_{w*}^{gr}.*

Proof. This is obvious by taking associated graded of Corollary 2.1.6. □

Definition 4.1.12. We also need the graded version of the Iwahori–Whittaker sheaves. Firstly, as before, we define the category $D_{\mathcal{IW},\mathrm{gr}}(\mathrm{Fl})$ via the formalism introduced in the paper[9] and Construction 2.4.5, with the construction of the character sheaf explained below.

For each $w \in {}^fW^{\mathrm{ext}}$, we define

$$\Delta_w^{\mathrm{gr}} = i_{w!}\Psi_w^*(\mathbb{AS})[\ell(w)]\langle\ell(w)\rangle, \nabla_w^{\mathrm{gr}} = i_{w*}\Psi_w^*(\mathbb{AS})[\ell(w)]\langle\ell(w)\rangle$$

We explain our notation here. We first consider $\mathbb{AS}$ as a sheaf over $(\mathbb{G}_a)_0/\mathbb{F}_q$, which by definition is a mixed sheaf over $\mathbb{G}_a$ pure of weight 0. Then, we let $\Psi^*(\mathbb{AS})$ be the desired character sheaf over I_0, which is also pure of weight 0 since Ψ is smooth. Then, note Ψ_w is also smooth, hence $\Psi_w^*(\mathbb{AS})$ is pure of weight 0. Hence, letting Δ_w^m and ∇_w^m be the corresponding mixed sheaves in $D_{I,\mathrm{mixed}}(\mathrm{Fl})$ given by

$$\Delta_w^m = i_{w!}\Psi_w^*(\mathbb{AS})[\ell(w)]\left(\frac{\ell(w)}{2}\right), \nabla_w^m = i_{w*}\Psi_w^*(\mathbb{AS})[\ell(w)]\left(\frac{\ell(w)}{2}\right)$$

which lives in weight ≤ 0 and ≥ 0 respectively. Then, taking associated graded sheaf at any of the process above gives our definition. Again we take the notations $\Delta_\lambda^{\mathrm{gr}}$ and $\nabla_\lambda^{\mathrm{gr}}$. In particular, we have

$$\Delta_e^m = \nabla_e^m, \Delta_e^{\mathrm{gr}} = \nabla_e^{\mathrm{gr}}$$

by an analogous proof as in Proposition 2.4.9. In particular, these sheaves are pure of weight 0. Moreover, we need the simple objects

$$\mathrm{IC}_w^{\mathrm{gr}} = i_{w!*}\Psi_w^*(\mathbb{AS})[\ell(w)]\langle\ell(w)\rangle$$

Definition 4.1.13. Finally, we define the graded version of Wakimoto sheaves. Namely, we define

$$J_\eta^{\mathrm{gr}} = j_{t_\mu *}^{\mathrm{gr}} *_I j_{t_\nu !}^{\mathrm{gr}}$$

for $\eta = \mu - \nu$, $\mu, \nu \in X_*(T)^+$.

Again, we also define the generalized Wakimoto sheaves

$$J_w^{\mathrm{gr}} = J_\lambda^{\mathrm{gr}} *_I j_{w_f *}^{\mathrm{gr}}$$

for $w = \lambda \cdot w_f \in W^{\mathrm{ext}}$, $\lambda \in X_*(T)$, $w_f \in W_f$.

Again, $J_\mu^{\mathrm{gr}} *_I J_\nu^{\mathrm{gr}} = J_{\mu+\nu}^{\mathrm{gr}}$, so the full subcategory $\mathbf{Perv}_{I,\mathrm{gr}}^{\mathrm{waki}}(\mathrm{Fl}) \subset \mathbf{Perv}_{I,\mathrm{gr}}(\mathrm{Fl})$, consisting of objects admitting a filtration by $J_\lambda^{\mathrm{gr}}\langle n\rangle$, is closed under convolution in $D_{I,\mathrm{gr}}(\mathrm{Fl})$.

Proposition 4.1.14. *We have $\mathcal{H} \simeq K^0\big(D_{I,\mathrm{gr}}(\mathrm{Fl})\big)$, with $H_w \mapsto (-1)^{\ell(w)}[j_{w*}^{\mathrm{gr}}]$.*

Proof. The proof is exactly the same as the proof of Proposition 2.5.6. □

Corollary 4.1.15. *For any $\lambda \in X_*(T)^+$, the class $[Z^{\mathrm{gr}}(V_\lambda)] \in \mathcal{H}$ is given by*

$$\sum_{\mu \in X_*(T)} \dim V_\lambda^{T^\vee = \mu} \cdot \theta_\mu$$

Proof. Also the same as Proposition 2.5.8. □

4.2. The construction of the graded functor

In this section, we will construct the functor Φ_{gr} in Conjecture 1.2.2.

Construction 4.2.1. The strategy of the construction would mostly follow from the previous construction. However, since we have an additional action of $\mathbb{G}_m \circlearrowright \widetilde{\mathcal{N}}^\vee$ given by t acting by $t : (b, x) \mapsto \big(b, t^{-2}x\big)$, under the embedding

$$\widetilde{\mathcal{N}}^\vee \hookrightarrow \mathcal{B}^\vee \times \mathfrak{g}^\vee$$

some of the construction must be modified.

Mainly, we do the following:

(1) Construct a functor

$$\Phi_{0,\,\mathrm{gr}} : \mathbf{Rep}(G^\vee \times T^\vee \times \mathbb{G}_m) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

(2) Use the de-equivariantization principle to upgrade it to a functor

$$\Phi_{1,\,\mathrm{gr}} : \mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

(3) Use the de-equivariantization principle to restrict it to a functor

$$\Phi_{2,\,\mathrm{gr}} : \mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

(4) Again pass to derived category, postcompose $\mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \hookrightarrow \mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})$ and construct

$$\Phi_{3,\,\mathrm{gr}} : D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \to D\big(\mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})\big)$$

(5) Compose with the functor $\mathrm{Av}^{\mathrm{gr}}_{\Psi}$

$$\Phi_{\mathcal{IW},\,\mathrm{gr}} : D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \to D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

Proposition 4.2.2. *The sheaf $Z^{\mathrm{gr}}(V) \in \mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})$ admits a filtration by the unshifted graded Wakimoto sheaves J^{gr}_λ, with the multiplicity of J^{gr}_λ given by* $\dim V^{T^\vee = \lambda}$.

Proof. We first mimic the proof of Proposition 2.3.1 to see that $Z^{\mathrm{gr}}(V)$ admits a filtration by the grade-shifted Wakimoto sheaves $J^{\mathrm{gr}}_\lambda\langle n\rangle$. Now let X be any graded convolution exact perverse sheaf. We apply Lemma 2.3.4, and take S to be the set in the lemma for X. We write $S = \{\lambda_i w_i\}$ such that $\lambda_i \in X_*(T)$ and $w_i \in W_f$. Then, we choose a coweight $\nu_0 \ll 0 \in X_*(T)$ such that $-(\nu_0 + \lambda_i)$ is strictly dominant. This implies that

$$J^{\mathrm{gr}}_{t_{-\nu_0}\cdot s} = j^{\mathrm{gr}}_{t_{-\nu_0}\cdot s!}$$

By definition of S, we have

$$\mathrm{Supp}\left(j^{\mathrm{gr}}_{(t_{-\nu_0})!} *_I X\right) \subset \bigsqcup_{s\in S} \mathrm{Fl}_{t_{-\nu_0}\cdot s}$$

Then, since graded perverse sheaves are generated (as a triangulated category) by $L_w\langle n\rangle$ (for all n), and by induction on $\ell(w)$ such sheaves are generated by $j_{w!}\langle n\rangle[m]$ ($m \geq 0$), we obtain

$$j^{\mathrm{gr}}_{t_{-\nu_0}!} *_I X \in \mathrm{Span}\{j^{\mathrm{gr}}_{w!}[i]\langle j\rangle \mid i \geq 0, j \in \mathbb{Z}\}$$

hence by support reasons

$$j^{\mathrm{gr}}_{t_{-\nu_0}!} *_I X \in \mathrm{Span}\Big\{j^{\mathrm{gr}}_{t_{-\nu_0}\cdot s!}[i]\langle j\rangle \mid i \geq 0, j \in \mathbb{Z}, s \in S\Big\}$$

By the equality $J^{\mathrm{gr}}_{t_{-\nu_0}\cdot s} = j^{\mathrm{gr}}_{t_{-\nu_0}\cdot s!}$, we have

$$J^{\mathrm{gr}}_{-\nu_0} *_I X \in \mathrm{Span}\Big\{J^{\mathrm{gr}}_{t_{-\nu_0}\cdot s}[i]\langle j\rangle \mid i \geq 0, j \in \mathbb{Z}, s \in S\Big\}$$

Hence, by the monoidality of $\Phi_{0,\,\mathrm{gr}}$, for any $\nu \in X_*(T)$ we have

$$J^{\mathrm{gr}}_{\nu} *_I X \in \mathrm{Span}\Big\{J^{\mathrm{gr}}_{t_\nu\cdot s}[i]\langle j\rangle \mid i \geq 0, j \in \mathbb{Z}, s \in S\Big\}$$

Thus, choose ν such that $\nu + \lambda_i$ are dominant for all i, we obtain

$$J^{\mathrm{gr}}_{\nu} *_I X \in \mathrm{Span}\Big\{j^{\mathrm{gr}}_{t_\nu\cdot s*}[i]\langle j\rangle \mid i \geq 0, j \in \mathbb{Z}, s \in S\Big\}$$

Since the left hand side is already perverse, we may remove the shift $[i]$ on the right. Hence $J^{\mathrm{gr}}_{\nu} *_I X$ admits a filtration by $J^{\mathrm{gr}}_{w}\langle j\rangle$, and by convolving with $J^{\mathrm{gr}}_{-\nu}$ on both sides, we obtain the Wakimoto filtration for X. Taking $X = Z^{\mathrm{gr}}(V)$ gives the first part of the claim.

Then, passing to K-groups, we have by Proposition 2.5.8 (its graded version of course) that the only objects appearing in the filtration are the objects $J^{\mathrm{gr}}_{\lambda}$, with multiplicity $\dim V^{T^\vee=\lambda}$. □

Definition 4.2.3. We define $\mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$ to be the category of graded sheaves with a filtration by sheaves $J^{\mathrm{gr}}_{\lambda}\langle n\rangle$.

Construction 4.2.4. We first construct

$$\Phi_{0,\,\mathrm{gr}} : \mathbf{Rep}(G^\vee \times T^\vee \times \mathbb{G}_m) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

Similar to the previous Φ_0, we define

$$\Phi_{0,\,\mathrm{gr}}\Big(V_\lambda \otimes \big(\overline{\mathbb{Q}}_\ell\big)_\eta \otimes \big(\overline{\mathbb{Q}}_\ell\big)_n\Big) \mapsto Z^{\mathrm{gr}}_\lambda *_I J^{\mathrm{gr}}_\mu\langle -n\rangle$$

Here, we define $Z^{\mathrm{gr}}_\lambda := \mathrm{gr}\Big(Z\Big(L^{\mathrm{Gr},\,\mathrm{gr}}_\lambda\Big)\Big)$, where $L^{\mathrm{Gr},\,\mathrm{gr}}_\lambda := j^{\mathrm{Gr}}_{\lambda!*}\overline{\mathbb{Q}}_\ell[\ell(\lambda)]\langle\ell(\lambda)\rangle$.

Again, this functor is monoidal, since Z^{gr} is central and monoidal by the fact proven in Theorem 2.2.1 for Z^m (the mixed version). Moreover, the functor

$$V_\lambda \mapsto L_\lambda^{\mathrm{Gr,\,gr}}$$

is symmetric monoidal by the finite field version of the geometric Satake equivalence (see Theorem 5.5.12[9]; here our group G_0 is split, so ${}^L G_0 = G_0 \times \Gamma$, and we can omit the second factor to obtain the desired functor), so $V_\lambda \mapsto L_\lambda^{\mathrm{Gr},m}$ is symmetric monoidal. Hence, by the compatibility of convolution with the functor gr, we have the above claim. Then, since Z_λ^{gr} is central and convolution exact, it admits a Wakimoto filtration, so the functor is indeed well-defined.

Construction 4.2.5. We then construct

$$\Phi_{1,\,\mathrm{gr}} : \mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee \times \mathbb{G}_m}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}_{I,\,\mathrm{gr}}^{\mathrm{waki}}(\mathrm{Fl})$$

Before the construction of this functor, we first construct the graded highest weight line $b_\lambda^{\mathrm{gr}} : Z_\lambda^{\mathrm{gr}} \to j_{t_\lambda *}^{\mathrm{gr}} = J_\lambda^{\mathrm{gr}}$. Here, the graded highest weight line exists because $\pi_* : \mathrm{Fl}_{t_\lambda} \to \mathrm{Gr}_\lambda$ is open dense, so the stalk $j_w^*(Z_\lambda^{\mathrm{gr}})$ is equal to the stalk of $L_\lambda^{\mathrm{Gr,\,gr}}$ in the graded setting, by Theorem 2.2.1(4), since the isomorphism $\pi_*(Z(-)) = \pi_!(Z(-)) = \mathrm{id}$ preserves Frobenius weights. Thus we obtain a morphism $j_w^*(Z_\lambda^{\mathrm{gr}}) \to \overline{\mathbb{Q}}_\ell[\ell(\lambda)]\langle \ell(\lambda)\rangle$, and taking adjunction gives the graded highest weight line $b_\lambda^{\mathrm{gr}} : Z_\lambda^{\mathrm{gr}} \to j_{t_\lambda *}^{\mathrm{gr}}$.

Analogous to the previous construction, we may construct a functor

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}\left(\overline{G^\vee/U^\vee}\right) \to \mathbf{Perv}_{I,\,\mathrm{gr}}^{\mathrm{waki}}(\mathrm{Fl})$$

via sending the highest weight line $B_\lambda : V_\lambda \otimes \mathcal{O} \to \mathcal{O}(\lambda)$ to the line $b_\lambda^{\mathrm{gr}} : Z_\lambda^{\mathrm{gr}} \to j_{t_\lambda *}^{\mathrm{gr}}$. Note that $\mathbb{G}_m$ does not act on the part $\overline{G^\vee/U^\vee}$, so this part remains untouched. For the other part, we need a functor

$$\mathrm{Coh}_{\mathrm{fr}}^{G^\vee \times \mathbb{G}_m}(\mathfrak{g}^\vee) \to \mathbf{Perv}_{I,\,\mathrm{gr}}^{\mathrm{waki}}(\mathrm{Fl})$$

since $T^\vee$ does not act on $\mathfrak{g}^\vee$. So this time, we need an $(G^\vee \times \mathbb{G}_m)$-equivariant morphism $\mathfrak{g}^\vee \to A$. Since $\mathbb{G}_m \circlearrowleft \mathfrak{g}^\vee$ by $t : x \mapsto t^{-2}x$, we can regard $\mathcal{O}(\mathfrak{g}^\vee)$ as the graded $G^\vee$-equivariant algebra $\overline{\mathbb{Q}}_\ell\left[(\mathfrak{g}^\vee)^*\right]$ with elements in $(\mathfrak{g}^\vee)^*$ of degree 2. Then, an $(G^\vee \times \mathbb{G}_m)$ -equivariant morphism $\mathcal{O}(\mathfrak{g}^\vee) \to A$ is a graded $G^\vee$-equivariant morphism $\mathcal{O}(\mathfrak{g}^\vee) \to A$, which is equivalent to a $G^\vee$-equivariant map of $\overline{\mathbb{Q}}_\ell$-vector spaces

$$(\mathfrak{g}^\vee)^* \to A_2$$

where A_2 denotes the piece of grade 2 of A. Then, this by definition corresponds to an element $x \in (\mathfrak{g}^\vee \otimes A_2)^{G^\vee}$. Then, using the functor of points

$$\begin{aligned}\mathfrak{g}^\vee \otimes A &= \ker(G(A[\varepsilon]) \to G(A)) \\ &= \{\text{functions } f : \mathcal{O}(G) \to A \text{ which is a derivation at } v_e \text{ (evaluation at } e)\} \\ &= \{\text{tensor derivations of} - \otimes A : \mathbf{Rep}\, G^\vee \to \mathbf{Mod}_A\}\end{aligned}$$

where derivation at v_e means $m^* f = f \otimes v_e + v_e \otimes f \in \mathcal{O}(G \times G)$. the final equivalence is given by the composition

$$V \otimes A \to V \otimes \mathcal{O}(G) \otimes A \to V \otimes A \otimes A \to V \otimes A$$

the first arrow is given by the comodule structure, the second by the map $f : \mathcal{O}(G) \to A$, and the final one by multiplication in A. Thus, an element in $(\mathfrak{g}^\vee \otimes A_2)^{G^\vee}$ corresponds to a $G^\vee$-equivariant tensor endomorphism of weight 2, since in the above composition the first map is of weight 0, the second of weight 2, and the final map of weight 0.

Hence, for a map

$$\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}_{\mathrm{fr}}(\mathfrak{g}^\vee) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

we need a weight 2 tensor derivation of the functor Z^{gr}, which is exactly the mixed version of the nilpotent endomorphism $M^m : Z^m(-) \to Z^m(-)(-1)$ in Theorem 2.2.2. Here, we must write it as $M^{\mathrm{gr}} : Z^{\mathrm{gr}}(-) \to Z^{\mathrm{gr}}(-)\langle -2 \rangle$.

Putting the two constructions together, we have constructed

$$\Phi_{1,\,\mathrm{gr}} : \mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee\right) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

Construction 4.2.6. Now we construct

$$\Phi_{2,\,\mathrm{gr}} : \mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl})$$

This again corresponds to the condition $b^{\mathrm{gr}} \circ M^{\mathrm{gr}}_{Z^{\mathrm{gr}}_\lambda} = 0$. This follows from $b \circ M_{Z_\lambda} = 0$, since the functor $\mathrm{oblv}_{\mathrm{gr}} : D_{I,\,\mathrm{gr}}(\mathrm{Fl}) \hookrightarrow D_I(\mathrm{Fl})$ is faithful.

Construction 4.2.7. We now construct

$$\Phi_{3,\,\mathrm{gr}} : D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \to D\big(\mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})\big)$$

By a graded analogue of Lemma 2.1.9, there exists an associated graded functor

$$\mathrm{gr}_{\mathrm{waki,\,gr}} : \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \to \mathbf{Rep}(T^\vee \times \mathbb{G}_m)$$

Since we have identified the associated graded in Proposition 4.2.2, we see that the composition

$$\mathbf{Rep}(G^\vee) \xrightarrow{Z^{\mathrm{gr}}} \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \xrightarrow{\mathrm{gr}_{\mathrm{waki,\,gr}}} \mathbf{Rep}(T^\vee \times \mathbb{G}_m)$$

is isomorphic to the restriction to $T^\vee$ and tensoring with the trivial $\mathbb{G}_m$-representation.

Then, we follow the previous proof of Proposition 3.2.10 and prove that the composition

$$\mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \xrightarrow{\Phi_{2,\,\mathrm{gr}}} \mathbf{Perv}^{\mathrm{waki}}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \xrightarrow{\mathrm{gr}_{\mathrm{waki,\,gr}}} \mathbf{Rep}(T^\vee \times \mathbb{G}_m)$$

is isomorphic to the pullback of coherent sheaves along the point $\iota : (1,0) \hookrightarrow \widehat{\mathcal{N}}^\vee_{\mathrm{aff}}$, this time regarding $(1,0)$ as a $T^\vee \times \mathbb{G}_m$ fixed point (note $\mathbb{G}_m$ acts purely on the fibers by scaling, so fixes 0). Again, we have the precomposition

$$\mathbf{Rep}(G^\vee \times T^\vee \times \mathbb{G}_m) \to \mathbf{Rep}(T^\vee \times \mathbb{G}_m)$$

given by the diagonal inclusion $T^\vee \hookrightarrow G^\vee \times T^\vee$ and the identity on $\mathbb{G}_m$, resulting in a map

$$T^\vee \times \mathbb{G}_m \hookrightarrow G^\vee \times T^\vee \times \mathbb{G}_m$$

So the upgrade

$$\mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \mathbf{Rep}(T^\vee \times \mathbb{G}_m)$$

corresponds to an $(G^\vee \times T^\vee \times \mathbb{G}_m)$-equivariant morphism $\mathcal{O}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) \to \overline{\mathbb{Q}}_\ell$. Again, it suffices to determine where $\mathrm{gr}_{\mathrm{waki,\,gr}}$ send the morphisms $M^{\mathrm{gr}}_{Z^{\mathrm{gr}}_\lambda} : Z^{\mathrm{gr}}_\lambda \to Z^{\mathrm{gr}}_\lambda$ and the highest weight line $b^{\mathrm{gr}}_\lambda : Z^{\mathrm{gr}}_\lambda \to j^{\mathrm{gr}}_{t_\lambda *}$ to. This time, $\mathrm{gr}_{\mathrm{waki,\,gr}} M^{\mathrm{gr}}_{Z^{\mathrm{gr}}_\lambda} = 0$, since it is sent by $\mathrm{oblv}_{\mathrm{gr}}$ to $\mathrm{gr}_{\mathrm{waki}} M_{Z_\lambda} = 0$, and $\mathrm{oblv}_{\mathrm{gr}}$ is faithful.

Then, the image of $\mathrm{gr}_{\mathrm{waki,\,gr}} b^{\mathrm{gr}}_\lambda$ agrees with $\mathrm{gr}_{\mathrm{waki}} b_\lambda$ since $\mathbb{G}_m$ does not act on $\overline{G^\vee/U^\vee}$. Hence, this composition again agrees with restriction at $\iota : (1,0) \hookrightarrow \widehat{\mathcal{N}}^\vee_{\mathrm{aff}}$.

Again, by smoothness of $\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}$, we have

$$K\left(\mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right)\right) = \mathrm{Perf}^{G^\vee \times T^\vee \times \mathbb{G}_m}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right) = D\mathrm{Coh}^{G^\vee \times T^\vee \times \mathbb{G}_m}_{\mathrm{fr}}\left(\widehat{\mathcal{N}}^\vee_{\mathrm{aff}}\right)$$

Thus taking homotopy categories, we have a functor

$$DCoh_{\mathrm{fr}}^{G^\vee\times T^\vee\times\mathbb{G}_m}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right) \to K\big(\mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})\big) \to D\big(\mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})\big)$$

Again, we have

$$DCoh^{G^\vee\times T^\vee\times\mathbb{G}_m}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)/DCoh^{G^\vee\times T^\vee\times\mathbb{G}_m}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)_{\partial\widehat{\mathcal{N}}^\vee} = DCoh^{G^\vee\times T^\vee\times\mathbb{G}_m}\left(\widehat{\mathcal{N}}^\vee\right)$$

and since the restriction ι^* kills $DCoh^{G^\vee\times T^\vee\times\mathbb{G}_m}\left(\widehat{\mathcal{N}}_{\mathrm{aff}}^\vee\right)_{\partial\widehat{\mathcal{N}}^\vee}$ because $(1,0)\notin\partial\widehat{\mathcal{N}}^\vee$. Thus we obtain a functor

$$\Phi_{3,\,\mathrm{gr}} : DCoh^{G^\vee\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \simeq DCoh^{G^\vee\times T^\vee\times\mathbb{G}_m} \to D\big(\mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl})\big)$$

Construction 4.2.8. We now give

$$\Phi_{\mathcal{IW},\,\mathrm{gr}} : DCoh^{G^\vee\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \to D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

We first recall the construction of the functor Av_Ψ. It is given by

$$\mathcal{F} \mapsto \Delta_e *_I \mathcal{F} : \mathbf{Perv}_I(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW}}(\mathrm{Fl})$$

We upgrade this to a functor

$$\mathrm{Av}_\Psi^{\mathrm{gr}} : \mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

Firstly, recall Δ_e^{gr} is pure of weight 0. Then, consider the convolution

$$\Delta_0^{\mathrm{gr}} *_I (-) : D_{I,\,\mathrm{gr}}(\mathrm{Fl}) \to D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

If for some mixed perverse sheaf $\mathcal{F}$ is such that $\Delta_0^{\mathrm{gr}} *_I \mathcal{F}$ is not perverse, by considering the diagram

$$\begin{tikzcd} D_{I,\,\mathrm{gr}}(\mathrm{Fl}) \arrow[r] \arrow[d] & D_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl}) \arrow[d] \\ D_I(\mathrm{Fl}) \arrow[r] & D_{\mathcal{IW}}(\mathrm{Fl}) \end{tikzcd}$$

we obtain a perverse sheaf $\mathcal{F}\in D_I(\mathrm{Fl})$ such that $\mathrm{Av}_\Psi(\mathcal{F})$ is not perverse, since $\mathrm{oblv}_{\mathrm{gr}}$ preserves the perverse t-structure by Theorem 4.1.6. This contradicts Proposition 2.4.10. Hence, we obtain a functor

$$\mathrm{Av}_\Psi^{\mathrm{gr}} = \Delta_0^{\mathrm{gr}} *_I (-) : \mathbf{Perv}_{I,\,\mathrm{gr}}(\mathrm{Fl}) \to \mathbf{Perv}_{\mathcal{IW},\,\mathrm{gr}}(\mathrm{Fl})$$

whose image under $\mathrm{oblv}_{\mathrm{gr}}$ is Av_Ψ. Hence, by composing, we obtain

$$\Phi_{\mathcal{JW},\,\mathrm{gr}} : D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \to D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})$$

In particular, the following diagram commutes

$$\begin{tikzcd}
D\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \arrow[r, "\Phi_{\mathcal{JW},\,\mathrm{gr}}"] \arrow[d, "\mathrm{oblv}_{\mathbb{G}_m}"'] & D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl}) \arrow[d, "\mathrm{oblv}_{\mathrm{gr}}"] \\
D\mathrm{Coh}^{G^\vee}\left(\widetilde{\mathcal{N}}^\vee\right) \arrow[r, "\Phi_{\mathcal{JW}}"] & D_{\mathcal{JW}}(\mathrm{Fl})
\end{tikzcd}$$

4.3. Proof of the graded equivalence

With the help of the commutative diagram in Construction 4.2.8, the final equivalence is relatively easy, mainly building on Theorem 4.1.5 and Theorem 2.4.1.

Lemma 4.3.1. *For simple objects $\mathcal{F}, \mathcal{G} \in \mathbf{Perv}_{\mathrm{gr}}(X)$, if $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}) = \mathrm{oblv}_{\mathrm{gr}}(\mathcal{G})$, then we have $\mathcal{F} = \mathcal{G}\langle n\rangle$ for some n.*

Proof. We apply Theorem 4.1.5, and obtain

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}\langle n\rangle) = \mathrm{Hom}\big(\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}), \mathrm{oblv}_{\mathrm{gr}}(\mathcal{G})\big)$$

Since $\mathcal{F}$ and $\mathcal{G}$ are simple, so is $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F})$ and $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{G})$. Hence, we have RHS $= \overline{\mathbb{Q}}_\ell$ a 1-dimensional vector space. Thus, exactly one direct summand on LHS is non-trivial, and is of dimension 1. Assume $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}\langle n\rangle) = \overline{\mathbb{Q}}_\ell$. Then since $\mathcal{G}$ is simple, so is $\mathcal{G}\langle n\rangle$. Thus we have constructed a non-trivial morphism between simple objects, hence gave an isomorphism $\mathcal{F} \simeq \mathcal{G}\langle n\rangle$. □

Proposition 4.3.2. *The sheaves $\mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})\langle n\rangle$ generates the category $D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})$ as a triangulated category.*

Proof. Firstly, we recall that $\mathbf{Perv}_{\mathcal{JW}}(\mathrm{Fl})$ is generated by IC_λ as a triangulated category. Then, for any sheaf $\mathcal{F} \in \mathbf{Perv}_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})$, we consider its filtration $F^\bullet\mathcal{F}$ with associated graded simple sheaves in $\mathbf{Perv}_{\mathrm{gr}}(\mathrm{Fl})$. Then, taking $\mathrm{oblv}_{\mathrm{gr}}(F^\bullet\mathcal{F})$, we obtain a filtration of $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F})$ with associated graded simple objects in $\mathbf{Perv}(\mathrm{Fl})$. Then, since we have $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}) \in \mathbf{Perv}_{\mathcal{JW}}(\mathrm{Fl})$, the associated graded simple objects must be of the form IC_λ.

Now, notice $\mathrm{oblv}_{\mathrm{gr}}(\mathrm{IC}_\lambda^{\mathrm{gr}}) = \mathrm{IC}_\lambda$, so the associated graded piece of $\mathrm{gr}_F^\bullet \mathcal{F}$ satisfies

$$\mathrm{oblv}_{\mathrm{gr}}(\mathrm{gr}_F^\bullet \mathcal{F}) = \mathrm{IC}_\lambda$$

so by Lemma 4.3.1 we have $\mathrm{gr}_F^\bullet \mathcal{F} = \mathrm{IC}_\lambda^{\mathrm{gr}}\langle n\rangle$ for some n. Hence, the sheaves $\mathrm{IC}_\lambda^{\mathrm{gr}}\langle n\rangle$ generates $\mathbf{Perv}_{\mathcal{IW},\mathrm{gr}}(\mathrm{Fl})$ as a finite length Abelian category, and hence generates $D_{\mathcal{IW},\mathrm{gr}}(\mathrm{Fl})$. Then, this implies that the standard objects $\Delta_\lambda^{\mathrm{gr}}\langle n\rangle$ and costandard objects $\nabla_\lambda^{\mathrm{gr}}\langle n\rangle$ generate the category by induction on lengths.

Then, we prove the asserted generation. We temporarily write w_λ for the element in W^{ext} which is of minimal length in the coset $W_f \cdot t_\lambda$. Firstly by definition we have

$$\mathrm{oblv}_{\mathrm{gr}}(\mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})\langle n\rangle) = \mathrm{Av}_\Psi(J_\lambda)$$

Thus $\mathrm{oblv}_{\mathrm{gr}}\big(i_w^! \mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})\big) = i_w^! \mathrm{Av}_\Psi(J_\lambda)$, so we have the support of $\mathrm{Av}_\Psi(J_\lambda) \subset \overline{\mathrm{Fl}^\lambda}$ and its stalk at the open strata is of rank 1. We prove that these objects generate the standard objects. Again we apply induction on lengths. For $\ell(w_\lambda) = 0$, we have $\mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}}\langle n\rangle) = \Delta_e^{\mathrm{gr}}\langle n\rangle$. Then, suppose these sheaves generate $\Delta_{w_\mu}^{\mathrm{gr}}\langle n\rangle$ for all n and $\ell(w_\mu) < k$. Then for some λ such that w_λ is of length k, we consider the triangle

$$K \to \Delta_{w_\lambda}^{\mathrm{gr}}\langle n_0\rangle \to \mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})$$

where the second arrow is obtained from applying the adjunction $i_{w_\lambda !} \dashv i_{w_\lambda}^!$ to the isomorphism of stalks $\Psi_w^*(\mathbb{AS})[\ell(w_\lambda)]\langle \ell(w_\lambda) + n_0\rangle \to i_{w_\lambda}^! \mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})$ (we do not yet know the exact weight of the shifted local system $\mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})$, so we need such a choice of n_0). Thus, we have K supported in $\overline{\mathrm{Fl}^\lambda} - \mathrm{Fl}^\lambda$, so generated by the standards $\Delta_\mu^{\mathrm{gr}}\langle m\rangle$ with $\ell(w_\mu) < k$. But, by our induction hypothesis, $\Delta_\mu^{\mathrm{gr}}\langle m\rangle$'s are generated by $J_\mu^{\mathrm{gr}}\langle m\rangle$'s, so K is generated by $J_\mu^{\mathrm{gr}}\langle m\rangle$'s. Hence, we arrive at a generation of $\Delta_{w_\lambda}^{\mathrm{gr}}\langle n_0\rangle$ by weight shifted Wakimoto sheaves, hence a generation of $\Delta_{w_\lambda}^{\mathrm{gr}}\langle n\rangle$ for arbitrary n.

Thus, we have the desired generation property. $\square$

Lemma 4.3.3. *Let X be any Noetherian stack with a $\mathbb{G}_m$-action. We have for any sheaves $\mathcal{F}, \mathcal{G} \in D\mathrm{Coh}^{\mathbb{G}_m}(X)$ the equality*

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D\mathrm{Coh}^{\mathbb{G}_m}(X)}(\mathcal{F}, \mathcal{G}\langle n\rangle) = \mathrm{Hom}_{D\mathrm{Coh}(X)}(\mathcal{F}, \mathcal{G})$$

Here $\mathcal{G}\langle n\rangle := \mathcal{G} \otimes k_{-n}$, where k_{-n} denotes the 1-dimensional $\mathbb{G}_m$ representation acting by $t : k \to k$, $1 \mapsto t^{-n}$.

Proof. We denote $p : X \to X/\mathbb{G}_m$ by the projection. Then, by the adjunction $p^* \dashv p_*$, we have

$$\mathrm{Hom}_{D\mathrm{Coh}(X)}(\mathcal{F},\mathcal{G}) := \mathrm{Hom}_{D\mathrm{Coh}(X)}(p^*\mathcal{F},p^*\mathcal{G}) = \mathrm{Hom}_{\mathrm{IndCoh}^{\mathbb{G}_m}(X)}(\mathcal{F},p_*p^*(\mathcal{G}))$$

where the left equality is by our abuse of notation forgetting the $\mathbb{G}_m$-equivariance, and the right equality is by adjunction. Then, by the projection formula, we have

$$p_*p^*\mathcal{G} = p_*(\mathcal{O}_X \otimes p^*\mathcal{G}) = p_*\mathcal{O}_X \otimes \mathcal{G}$$

However, since $X \to X/\mathbb{G}_m$ is a $\mathbb{G}_m$-torsor, we have

$$p_*\mathcal{O}_X = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{X/\mathbb{G}_m}\langle n\rangle$$

Finally, since $\mathcal{F}$ is a coherent complex, it is a compact object, so we have

$$\begin{aligned}
\mathrm{Hom}_{D\mathrm{Coh}(X)}(\mathcal{F},\mathcal{G}) &= \mathrm{Hom}_{\mathrm{IndCoh}^{\mathbb{G}_m}(X)}(\mathcal{F},p_*p^*(\mathcal{G})) \\
&= \mathrm{Hom}_{\mathrm{IndCoh}^{\mathbb{G}_m}(X)}\left(\mathcal{F},\bigoplus_{n\in\mathbb{Z}}\mathcal{G}\langle n\rangle\right) \\
&= \mathrm{Hom}_{\mathrm{IndCoh}^{\mathbb{G}_m}(X)}\left(\mathcal{F},\operatorname*{colim}_{N\to\infty}\bigoplus_{-N\le n\le N}\mathcal{G}\langle n\rangle\right) \\
&= \operatorname*{colim}_{N\to\infty}\mathrm{Hom}_{D\mathrm{Coh}^{\mathbb{G}_m}(X)}\left(\mathcal{F},\bigoplus_{-N\le n\le N}\mathcal{G}\langle n\rangle\right) \\
&= \operatorname*{colim}_{N\to\infty}\bigoplus_{-N\le n\le N}\mathrm{Hom}_{D\mathrm{Coh}^{\mathbb{G}_m}(X)}(\mathcal{F},\mathcal{G}\langle n\rangle) \\
&= \bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D\mathrm{Coh}^{\mathbb{G}_m}(X)}(\mathcal{F},\mathcal{G}\langle n\rangle)
\end{aligned}$$

Here we used that finite direct sums are finite products. □

Theorem 4.3.4. *There is a monoidal equivalence of categories*

$$\Phi_{\mathcal{JW},\,\mathrm{gr}} : D\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right) \simeq D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})$$

Proof. We first prove that this functor is fully faithful. Let $\mathcal{F},\mathcal{G} \in D\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right)$, and we wish to prove

$$\mathrm{Hom}_{D\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}\left(\widetilde{\mathcal{N}}^\vee\right)}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})}\left(\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{F}),\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{G})\right)$$

is an isomorphism. Then, by construction of $\Phi_{0,\,\mathrm{gr}}$, we have

$$\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{G})\langle n\rangle = \Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{G}\langle n\rangle)$$

This gives the following morphism

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}(\widetilde{\mathcal{N}}^\vee)}(\mathcal{F},\mathcal{G}\langle n\rangle) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})}\big(\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{F}),\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{G})\langle n\rangle\big)$$

By Construction 4.2.8, we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}(\widetilde{\mathcal{N}}^\vee)}(\mathcal{F},\mathcal{G}\langle n\rangle) \arrow[r, "\Phi_{\mathcal{JW},\,\mathrm{gr}}"] \arrow[d, "\mathrm{oblv}_{\mathbb{G}_m}"'] & \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D_{\mathcal{JW},\,\mathrm{gr}}(\mathrm{Fl})}\big(\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{F}),\Phi_{\mathrm{gr},\mathcal{JW}}(\mathcal{G})\langle n\rangle\big) \arrow[d, "\mathrm{oblv}_{\mathrm{gr}}"] \\
\mathrm{Hom}_{D\mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee)}(\mathcal{F},\mathcal{G}) \arrow[r, "\Phi_{\mathcal{JW}}"] & \mathrm{Hom}_{D_{\mathcal{JW}}(\mathrm{Fl})}(\Phi_{\mathcal{JW}}(\mathcal{F}),\Phi_{\mathcal{JW}}(\mathcal{G}))
\end{tikzcd}$$

The bottom arrow is an isomorphism by Theorem 2.4.1. The vertical arrows are isomorphisms by Theorem 4.1.5 and Lemma 4.3.3. Thus the top row is an isomorphism, hence restricting to isomorphisms in each degree. To sum up, the functor $\Phi_{\mathcal{JW},\,\mathrm{gr}}$ is fully faithful.

We then prove essential surjectivity. By Proposition 4.3.2, it suffices to show the image of $\Phi_{\mathcal{JW},\,\mathrm{gr}}$ contains $\mathrm{Av}_\Psi(J_\lambda^{\mathrm{gr}})\langle n\rangle$, which is evident, since

$$\Phi_{0,\,\mathrm{gr}}(V_0\otimes k_\lambda\otimes k_n) = \mathrm{Av}_\Psi^{\mathrm{gr}}(J_\lambda^{\mathrm{gr}})\langle -n\rangle$$

Thus this functor is an equivalence of categories! $\square$

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声明

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